AD 730508

☐ operations research and consulting services ☐ management information, planning and control systems ☐ economic and public policy studies ☐

MATHEMATICA INC.

195

Issuing Activity:

MATHEMATICA, Inc.
Bethesda, Md.
Princeton, N. J

ONR Contract Number:

N00014-68-C-0379

Sponsoring Activity:

Naval Analysis Programs
Office of Naval Research

Task Number:

NR 276-023

APPLICATIONS OF MATHEMATICAL ANALYSIS TO DETECTION PROBLEMS OF NAVAL STRATEGY (U)

by

John P. Mayberry
Francis M. Sand
Alan J. Truelove
John E. Walsh

September 29, 1969

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) MATHEMATICA, Inc. P. O. Box 92 Princeton, New Jersey 08540	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
Applications of Mathematical Analysis to Detection Problems of Naval Strategy (U)

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Final Report - March, 1968 to September, 1969.

5. AUTHOR(S) (First name, middle initial, last name)
Dr. John P. Mayberry
Francis M. Sand
Dr. Alan J. Truelove
Dr. John E. Walsh

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 29, 1969	185	4

8a. CONTRACT OR GRANT NO. N00014-68-C-0379	9a. ORIGINATOR'S REPORT NUMBER(S) F-6232
b. PROJECT NO. RR 003-11-01	
c. NR 276-023	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) None
d.	

10. DISTRIBUTION STATEMENT
Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Naval Analysis Programs (Code 462) Office of Naval Research Arlington, Virginia 22217

13. ABSTRACT

~~This~~ document is MATHEMATICA's final report~~ under Contract N00014-68-C-0379.~~ It describes the results of our mathematical research in the following areas:

A destroyer versus submarine search game; discrete two-person game theory with median payoff criterion; formulation and solution of games with incomplete information (I-games); formulation of a two-stage search and pursuit game; ~~and~~ general game theory and statistical research applicable to ASW problems.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Game Theory						
Median Games						
Anti-Submarine Warfare (ASW)						
Destroyer						
Submarine						
Search Games						
Mixed Strategies						
Statistics						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Security Classification

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CHAPTER I

INTRODUCTION

This report, MATHEMATICA's report F-6232, presents the results of research performed under ONR Contract N00014-68-C-0379, and Mod. P001 of that contract, during the period March 1968 to September 1969.

The objectives of the research have been:

(i) to identify actual operational ASW problems on which game-theoretic methods give convincing evidence that valuable insights can be provided;

(ii) to formulate mathematical problems which reflect the aspects of actual ASW problems which are believed to be critical for practical operational solutions;

(iii) to investigate the solvability of those mathematical problems;

(iv) to completely solve at least a sample of significant special cases of the above mathematical problems.

In particular, we have given major emphasis to the extension of research performed by MATHEMATICA under previous ONR contracts, so that the earlier models may include more operationally significant factors.

The remainder of this report consists of seven chapters which are summarized in the following.

Chapter II - A DESTROYER VERSUS SUBMARINE SEARCH GAME

The basic problem considered in this paper is the following:

An Evader E (enemy submarine) is at position O when he is "sighted" by a Pursuer P (submarine, surface ship, fixed-wing aircraft or helicopter). The fact of the sighting is known by F. Assuming that E has time to take evasive action, the problem to be solved is the determination of the path P must follow in order to maximize his probability of detecting E.

For this problem we allow both E and P to have a wide, but simplified set of strategies. E can choose a direction at random and adopt a fixed speed. P guesses E's speed and chooses his own speed and proceeds to O. He must select a spiral search path which originates at a point on his trajectory, this point being a function of his assumed speed of E.

The strategies for P are expressable as closed, algebraic expressions which enables us to compute them by hand. A numerical example is computed and show that for nearly all strategies for E. The resultant strategies for P are superior to those obtained by inconvenient and expensive computer techniques. It is conjectured that P's strategies are near optimal for all of E's strategies.

The results and extensions of this work include the computation of the probability of detection by multi-pursuer forces, estimating the effective detection radius of actual fleet exercises, sensitivity analyses of the problem parameters, strategies against multi-evader forces, and the determination of the optimal composition of mixed force ASW fleets.

Chapter III - DISCRETE TWO-PERSON GAME THEORY WITH MEDIAN PAYOFF CRITERION

The paper by John E. Walsh presents a basic definition of the median pay-

off criterion, as it might be applied to a two-person non-zero sum game (since the payoffs for each player are considered to be merely ordinal, and not strictly quantitative in the usual sense, the concept of "zero sum" should really be replaced by the notion of "strictly competitive"). The median-optimum strategies for each player under various circumstances are defined, and their existence is shown.

The appendix to Chapter III derives some further results, which are obtained by restricting the application of the median payoff criterion to zero-sum (i.e., strictly-competitive) two-person games. Algorithms for finding the solutions under these circumstances are suggested. An outline of the proof is also sketched, showing that if the payoffs in the two-person matrix game are not determinate, but are each sampled from a continuous probability distribution, then the median-optimum payoff value will be unique (in the main paper, uniqueness could not be assured). This approach may be valuable in cases where quantitative payoff information cannot be assured, but where preference orderings among the payoffs can be reliably provided.

Chapter IV - MIXED STRATEGIES IN PRACTICE

This expository paper develops an explanation of the practical usefulness of the concept of mixed, or random, behavior in a two person game -- in particular, in a two-person zero-sum game. It is the thesis of this paper that many of the common intuitive objections, to the use of mixed strategies in practice, are either incorrect or can be avoided by suitable methods of choosing the mixed strategies.

Chapter V - REJECTION OF DATA

This paper presents some statistical methods which we believe will be useful in situations where measurements are made which must be subsequently as-

signed to distributions. There are two real situations in which problems of this type arise, and towards which the results of this paper represent a partial solution. The first, and most important, is the distinction between the probability distribution of sonar returns when a real target is present, and the probability distribution of sonar returns when no real target is present. The second represents the distinction between the probability distribution of sonar returns when various physical mechanisms or transmission are operating. For example, detection ranges when bottom-bounce is present are quite different from detection ranges experienced when other phenomena are governing.

Chapter VI - A SIMPLE I-GAME

Chapter VII - THE GENERALIZATION AND SOLUTION OF THE SMALL I-GAME DETECTION MODEL

These papers represent the formulation, and the solution, of two families or simplified games of incomplete information (I-games). The I-game methodology is the result of an attempt to extend game theory and to apply it to situations where the physical background, i.e. the rules of the game, are not entirely known to both players. The examples treated here are too simplified to be of direct applicability, but are intended as prototypes for the development of more complex games of incomplete information whose results might be more directly applicable.

Chapter VIII - FORMULATION OF A GAME OF SEARCH AND [illegible]SUIT

This chapter describes a two stage game -- a searc[illegible]ollowed by a pursuit game. Search games have been treated for a considerable period of time by standard game-theoretical methods, while games of pursuit have also been extensively studied. This present formulation is an attempt to develop a more

realistic situation, because the outcome of the search stage is determined by the result predicted for the pursuit which begins when the search ends. It also provides some significant insights as to what represents reasonable behavior for a Pursuer and an Evader, if the result of the search will not be an immediate payoff, but an ultimate payoff determined by the result of the pursuit. Although this game has not been solved, we present qualitative geometric descriptions of the expected solution. We believe that additional computation will suffice to provide a quantitative specification of optimal behavior during the search stage of such a game.

In addition to the authors of the chapters of this report, valuable contributions have been made by Louis Auslander, Peter J. Kalman, and Michel L. Balinski. Some of the contents of this report were given at informal briefings, and we benefited from comments received. In this connection we would like to thank Dr. Paul L. Warnshuis, Jr., Naval Undersea Research and Development Center, San Diego; Dr. Calvin Sweat, Naval Undersea Research and Development Center, Pasedena; and Dr. William F. Lucas, the RAND Corporation.

The editors of this report were Alan J. Truelove and John Mayberry.

CHAPTER II

A DESTROYER VERSUS SUBMARINE SEARCH GAME

Alan J. Truelove

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A DESTROYER VERSUS SUBMARINE SEARCH GAME

0.0 Introduction and Summary of Results

The basic problem considered in this paper is the following: An Evader E (enemy submarine) is at position O when he is "sighted" by a Pursuer P (submarine, surface ship, fixed wing aircraft or helicopter). The fact of the sighting is known by E. Assuming that E has time to take evasive action, the problem to be solved is the determination of the path P must follow in order to maximize his probability of detecting E.

For this problem we allow both E and P to have a wide, but simplified set of strategies. E can choose a direction at random and adopt a fixed speed. P guesses E's speed and chooses his own speed and proceeds to O. He must select a spiral search path which originates at a point on his trajectory, this point being a function of his assumed speed of E.

The strategies for P are expressable as closed, algebraic expressions which enables us to compute them by hand. A numerical example is computed and show that for nearly all strategies for E. The resultant strategies for P are superior to those obtained by inconvenient and expensive computer techniques. It is conjectured that P's strategies are near optimal for all of E's strategies.

The results and extensions of this work include the computation of the probability of detection by multi-pursuer forces, estimating the effective detection radius of actual fleet exercises, sensitivity analyses of the problem parameters, strategies against multi-evader forces, and the determination of the optimal composition of mixed force ASW fleets.

0.1 Statement of the Problem

We consider the following model in search theory (Fig. 1).

An Evader (E) is "sighted" at a point O, at time $t = 0$; we refer to this position-time combination as the "datum". Contact with E is then lost, because he takes evasive action, knowing he has been sighted.

We restrict E's strategy as follows:

(i) E chooses a direction at random

(ii) E adopts a fixed speed v, $0 \leq v \leq v_o$.

A pursuer (P) is stationed at distance D from O at time $t = 0$. He proceeds directly towards O at maximum speed, Q. P makes a guess at the speed v chosen by E. He stops at the point A where he would meet E, if E had in fact chosen speed v, and chosen the direction towards P; from point A, P describes an outward spiral (constant pitch, with center O), at his "search" speed q, which will generally be lower than his maximum speed. This outward spiral is such that it matches E's speed v exactly, that is, the radial component of P's velocity is precisely v. Then, if the guess v was exact, P is certain to meet E before he has completed a revolution about O.

Now suppose P has a detector, with "cookie-cutter" radius R. Then, even if P does not guess v exactly, he still has a chance of detecting E. The question discussed in this paper is, "What 'mix' of spirals should P adopt in order to maximize Pr(detection of E)?" We also consider the "dual" question, namely, "What mix of speeds should E adopt?"

E is assumed to know P's distance at $t = 0$, but not P's bearing. Both P and E know each other's maximum speeds.

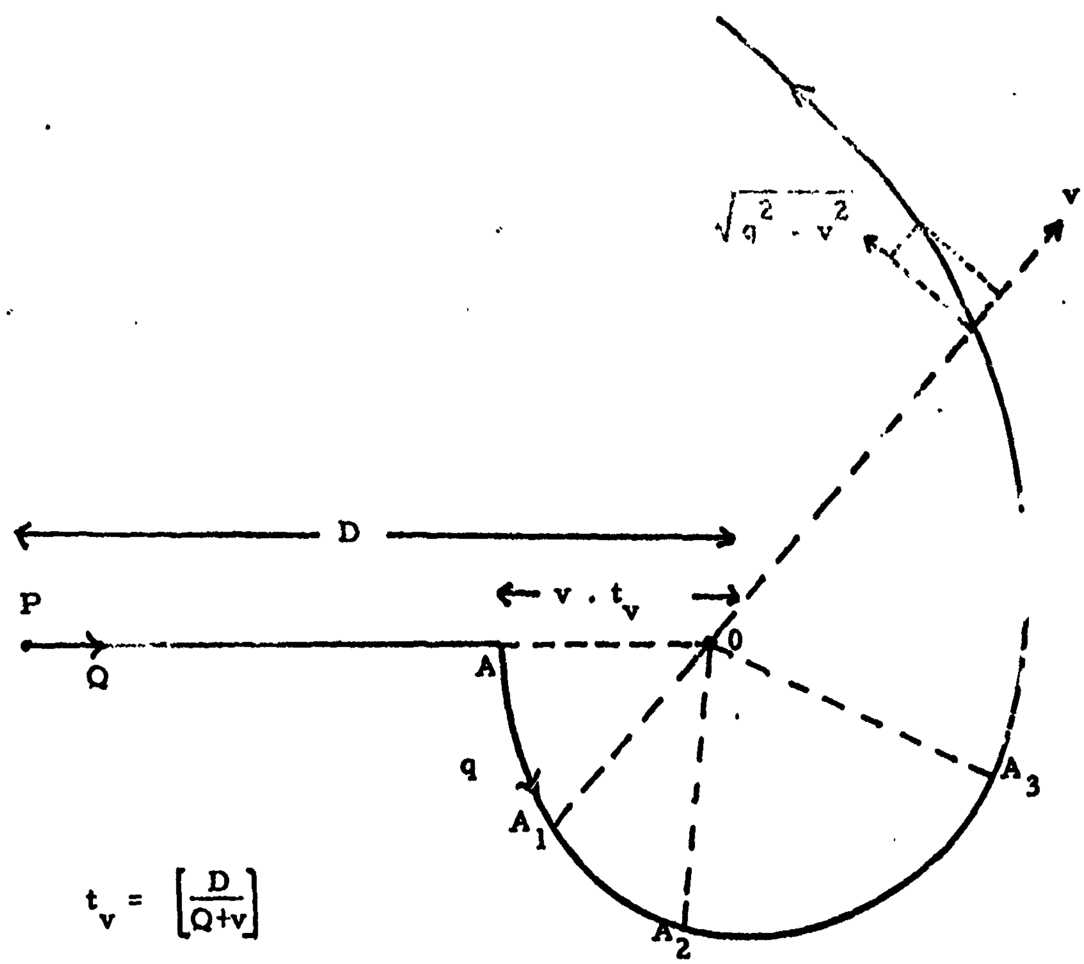


FIGURE 1 - P's Spiral Strategy, 'v'.

[If E chooses speed v, he could be detected at (for example) points $A_1, A_2, A_3, \ldots$, depending on the direction E chooses]

0.2 Summary of Results

The "mix" of spirals (i.e., the corresponding probability density $g^*(v)$, of the choice of v) which is optimal for P is determined approximately. A closed form expression for $g^*(v)$ is found, the coefficients of which are functions of the physical parameters of the model, and the resulting Pr(detection) is calculated.

The optimal strategy for E is not found explicitly, but we give a heuristic argument showing that E's strategy is almost identical with the strategy obtained for P.

Since this general solution is only approximate, an exact solution for a particular case was computed, using linear programming. It was necessary to restrict P's strategy to a mix of four particular strategies. The results confirm that the approximate solution is very nearly as good as this (restricted) exact solution.

The strategy for P, $g^*(v)$, and the resulting Pr(detection), are graphed below. [Figs. 2 and 3]. E's maximum speed is taken as 1, so that P's speed is expressed in terms of this unit.

Fig. 2 - Strategies for the Pursuer, P

Key: $g^*(v)$ = 'Ad-hoc' strategy, using 4 parameters

$g^{**}(v)$ = Modified 'Ad-hoc' strategy, improving coverage at v=1

$g^*_{LP}(v)$ = Strategy produced by Linear programming method, using four parameters.

$g^*(v)$ = Probability density function for allocation of search effort to radius v of speed circle

Q = 2.0
q = 2.0
R = 0.05
d = 2.0

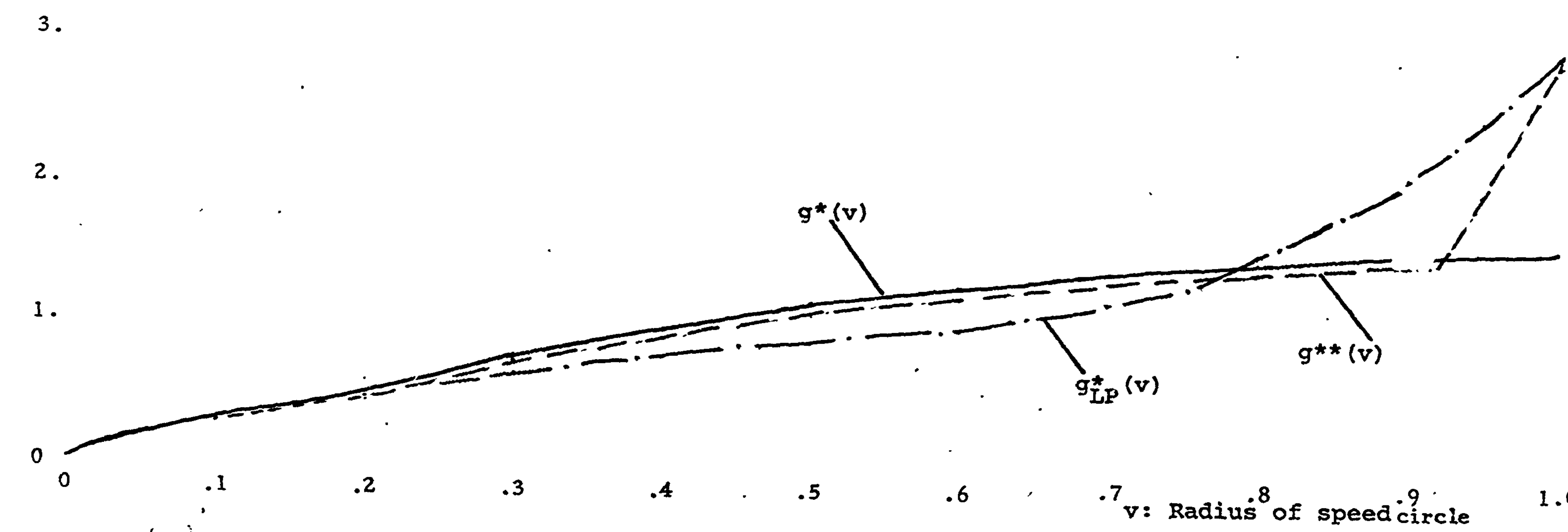
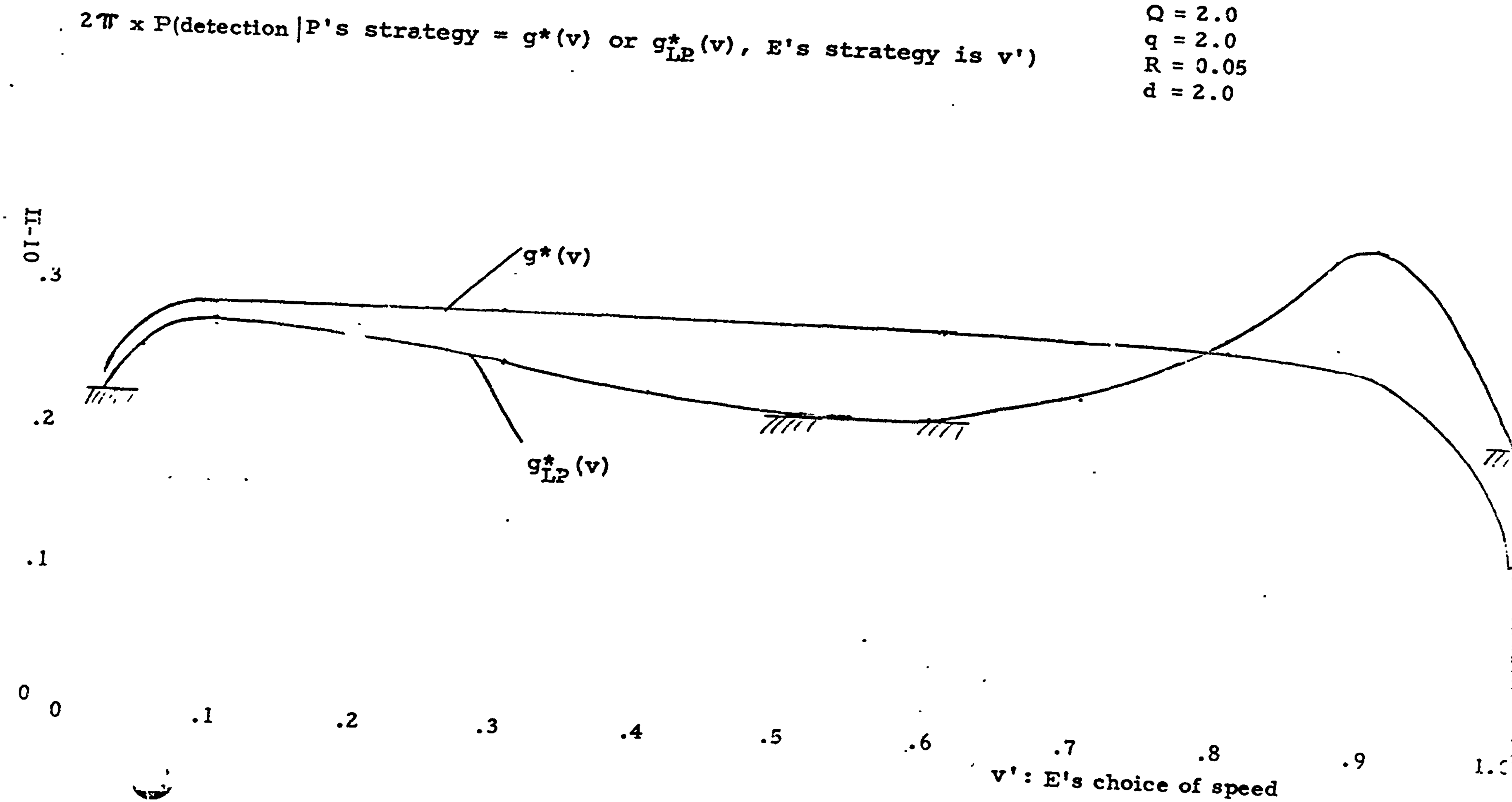


Fig. 3 - Probability of Detection

Key: $g^*(v)$ = 'Ad-hoc' strategy

$g^*_{LP}(v)$ = Strategy produced by Linear programming method



0.3 Comparison of Results with Danskin's Helicopter Search

Danskin (1), considered a Helicopter (P) versus Submarine (E) game, and introduced the 'speed circle' concept (see following section).

The differences between his model and ours are:

(a) The helicopter performs a 'dip' for a finite specified period of time, with a cookie-cutter detector.

(b) He assumes the helicopter takes zero time to fly from one dip-point to any other dip-point in the search.

Danskin's solution specifies identical strategies for the allocation of Helicopter search effort, and the allocation of Submarine choice of v , namely the $g_D(v)$ function shown below.

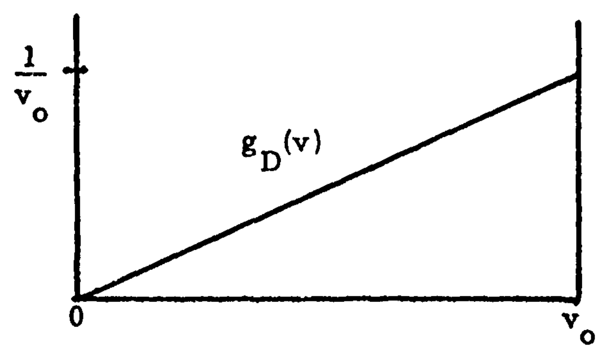


FIGURE 4 – Danskin's solution for both P and E , $g_D(v)$.

In our case, we agree (heuristically) that the strategies for P and E should be approximately identical, under reasonable assumptions on cookie-cutter range, initial distance D , and relative speeds (see section 10, 'A Smeared-Diagonal Zero-Sum Game').

However, in our problem P is constrained to move in a continuous path. There are two effects which tend to counteract each other. (In the description, we make use of the speed space transformation, described in the following section).

(a) P arrives at the rim of the speed circle when t, the elapsed time, is still small. His relative cookie-cutter radius is large, so search circles near the rim are more effective than in the helicopter case.

(b) By the time P gets to (or near) O, his cookie-cutter is small. On the other hand, he has more excess speed to use on transverse motion, since his required radial speed is small or zero, to match that assumed for E. Hence, he covers more ground.

Our recommended strategy $g^{**}(v)$, for the numerical case considered, does not differ too much from Danskin's solution $g_D(v)$. In fact it shows a slight "hump" for the middle range of v. (There is an anomalous spike at $v = 1$, the reason for which, explained later is to deal with the difficulty of covering values of v near the rim $v = 1$; this anomaly was not considered by Danskin, who assumed that the successive helicopter circles, having decreasing radius in speed space, could be packed together with sufficient accuracy.)

0.4 Uses of the Results and Indicated Further Work

The folk ing are possible uses of our single-speed search model in realistic situations:

(a) Computation of Pr(detection) by Multi-Unit Forces

In specified situations, e.g., 4 destroyers situated 10 miles away from the point of detection, it is possible to calculate the probability of detection, or, conversely, the probability that the submarine will escape.

(b) Estimation of Effective Detection Radius

Suppose that destroyer vs. submarine (or similar) games are observed, in situations which are reasonably close to those assumed in the model.

Then we may observe (over repeated trials) the probability of successful search and, in fact, the time and position of the final detection. From these data, we can then estimate certain parameters of the model, e.g. effective "cookie-cutter detection radius.

(c) Optimization of Equipment Parameter

The parameters of the problem could be varied to yield answers to the following:

(i) How sensitive is Pr(detection) to the ratio of maximum speeds, P to E ?

(ii) What is the effect of replacing the assumption that E knows P's distance D , by the assumption that E has an a priori distribution on D ?

(iii) Same as (ii), if E has only an a priori distribution on P's maximum speed Ω , and/or on P's search speed, q .

(d) Mixed Force Problems

Suppose that the Pursuer's force consists of two or more of the following:

Surface ships,
Pursuit submarines,
Helicopters (e.g., Aircraft carrier based),
Fixed-wing aircraft.

Using suitable mathematical programming techniques, we can determine the optimal allocation of portions of the search areas to the various force components.

We can also, for the various components, determine the utility of different initial distances, and approach speeds.

(e) Optimal Composition of Mixed Force ASW Fleets

For ASW fleets, it is better to have a few large mixed forces, or more, but smaller groups? (Assuming that initial contacts with Evaders are made randomly over the tactical area). Graphs showing Pr(detection) vs. Force size, with initial distance from the datum as a parameter, would be a start in solving this problem.

0.5 Remark on More Sophisticated Strategies

The Pursuer has been restricted to outward spirals, starting from some point on the line between his initial position and the datum while the Evader has been restricted to single direction, single speed strategies (following Danskin). The question naturally arises, whether significant improvement on either side could be obtained by using more sophisticated strategies.

First, since E knows that some interval will elapse before P can come within range, there is an argument for proceeding at maximum or high speed for most of this interval. A two-stage speed model is considered later in the paper, but no results have been obtained.

There seems no good reason for E to choose other than a fixed direction, assuming that he cannot detect P's search path. Zig-zagging, back-tracking, etc., would mean that less potential ground is covered, from P's point of view, at a greater speed.

Next we come to P's strategy. It is seen that the recommended mix of spirals results approximately in a "uniform" coverage of the speed circle, which is precisely the solution that Danskin obtained in the simpler game. Wherever E places himself in the speed circle, the probability of detection is the same. It is, heuristically, clear (as mentioned later on in section 10) that any optimal strategy not restricted to spirals for P must result in uniform coverage. It appears likely that our method is at least as economical as any other, i.e. that any other mix of strategies would result in a value of the

game (Pr(detection)) equal to or lower than the value we obtained here. If this is true, then our class of strategies is "complete" in the usual statistical sense.

1. Preliminaries

1.0 The Model

At time $t=0$, an Evader (E) is sighted at a point 0. The position/time combination being referred to as the 'datum'. E realizes he has been sighted, dives, and adopts the following strategy:

(i) E chooses a direction at random,

(ii) E chooses a fixed speed v, $0 \leq v \leq v_0$.

A Pursuer P is located at distance D from O ; E does not know E's speed or bearing. P knows E's maximum speed v_o ; E knows P's distance D , maximum and search speeds (see below) Q, q ; and each is aware that the other has this information.

P proceeds at maximum speed Q towards the datum point, and adopts, from some point onwards, with speed $q \leq Q$, a strategy chosen from a set to be defined.

P and E are blind to each other, except that if E comes within range R of P (the 'cookie-cutter' assumption), P is considered to detect E .

1.1 The 'Speed Circle' Transformation

It is convenient to introduce the following mathematical transformation. O is kept fixed.

The bearings of P and E from O are left unchanged. But radial distances from O are multiplied by the factor $1/t$, where t is the time elapsed since the datum. It will be seen that the position of E in this transformed picture (called 'speed space') will be a fixed point, somewhere in the circle center O , radius v_o, called the speed circle, and this point will be labelled 'v' .

Since P's radial component of velocity is v, his path will be represented by a circle radius v. In fact, if P continues his spiral (in real space) indefinitely, he will trace this circle over and over.

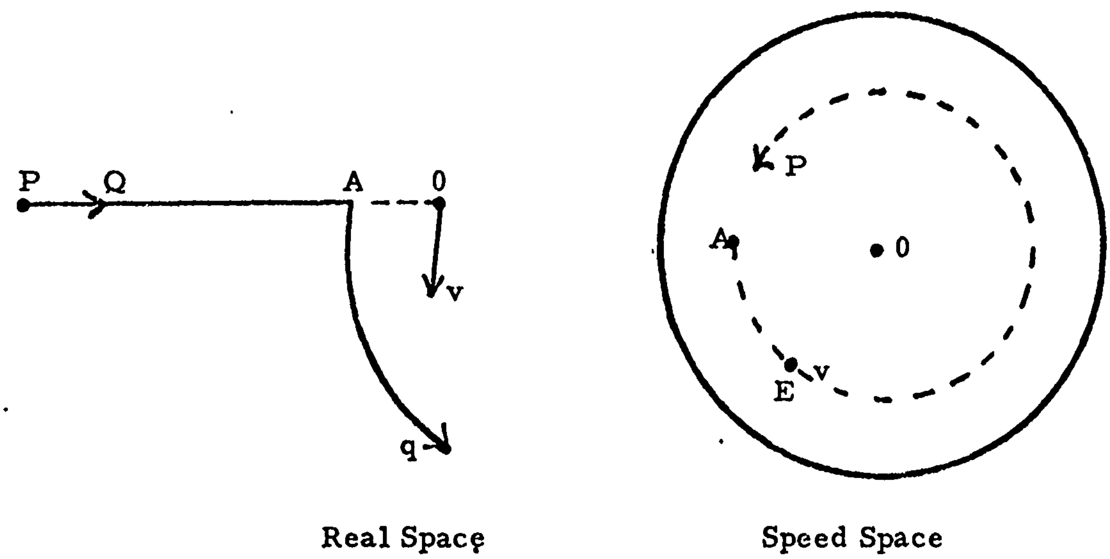


FIGURE 5 – Real and Speed Space

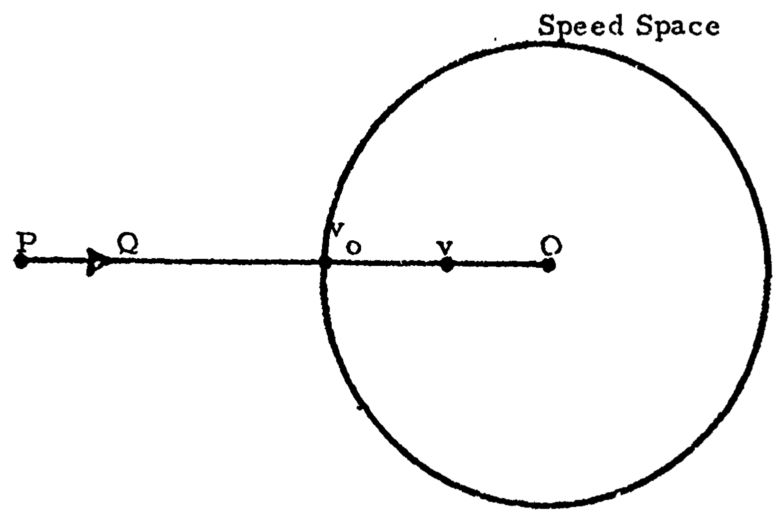
2. Pursuer's Speeds and Strategy Class S

Generally P will be assumed to have 2 available speeds:

The maximum speed of Q at which the detection probability is so small that it can be ignored and a search speed of q, at which the cookie-cutter assumption holds.

The class S of strategies for P will be defined as follows:

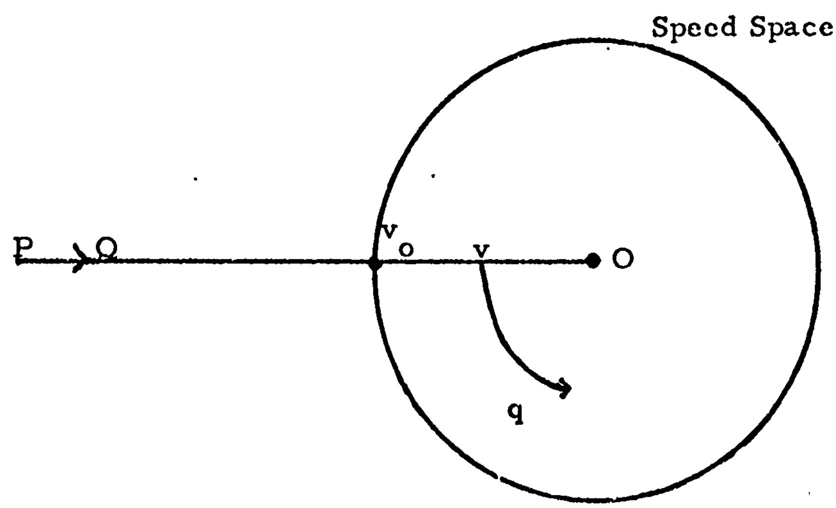
P proceeds at speed Q to some poi' t A which will correspond (we suppose) to a point v in the speeu circle arriving at time t_v.



(2.1) $$(Q + v)\, t_v = D \,,$$

(2.2) $$t_v = \frac{D}{Q + v} \quad .$$

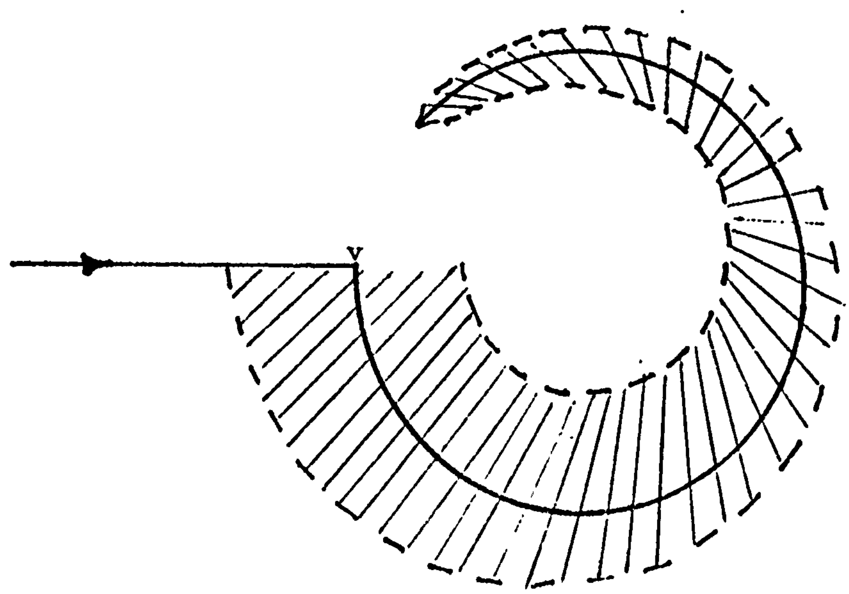
Next, P adopts (in real space) a spiral, contained in the class S, corresponding in speed space to a circle and steaming at 'search speed' q .



3, Area of Coverage.

In speed space, the radius of detection is R/t.

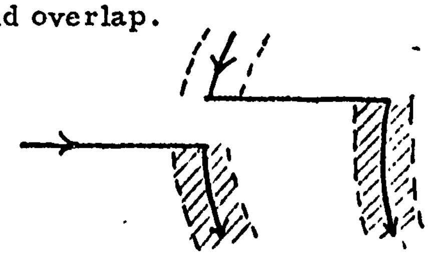
The area covered by P's search may be shown thus:



If the circle starts from point v, we will denote this shaded area by a_v.

Consider the fixed circle, in speed space, radius v'.

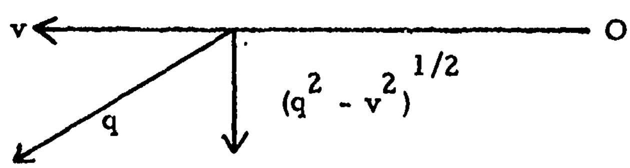
How much of the perimeter of this circle will we cover by a_v? If, in fact a_v still has significant width after a rotation of 2π, we suppose that the shaded area is, from then on, displaced radially just enough to avoid overlap.



This means that a coverage of the perimeter of more than $2\pi v$ will have meaning, as will become clear.

In speed space, the speed along the arc (radius v) is

$$\frac{(q^2 - v^2)^{1/2}}{t}$$



This is because in real space, P must steam radially away from O with component speed v to 'stay where it is' (like Alice) in speed space. This leaves a component $(q^2 - v^2)^{1/2}$ transversely which in speed space becomes $(q^2 - v^2)^{1/2}/t$.

The shaded area, a_v, will cover part of the arc at v', so long as

(3.2) $$\frac{R}{t} \geq |v - v'| ,$$

i.e., up to the time

(3.3) $$t_1 = R / |v - v'| .$$

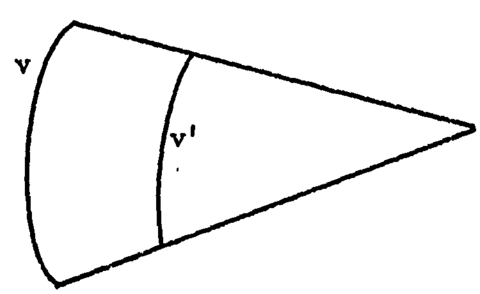
Distance travelled along the arc at v at time t_1 is

$$DA = \int_{t=t_v}^{t_1} \frac{(q^2 - v^2)^{1/2}}{t} dt = (q^2 - v^2)^{1/2} \log (t_1/t_v)$$

$$= (q^2 - v^2)^{1/2} \log \frac{R/|v - v'|}{D/(Q + v)}$$

using (2.2), (3.3).

How much of the arc at v' is 'covered'?



Obviously, $(D\Lambda).(\frac{v'}{v})$, whether $v' \geq v$ or $\leq v$.

In general, not all the arcs of radius v in the range $[0, v_0]$, will contribute to the coverage. In fact, for any contribution to be made, we must have

(3.4) $$t_1 > t_v .$$

Hence, using (2.2)

(3.5) $$\frac{R}{|v - v'|} > \frac{D}{Q + v} .$$

Let

(3.6) $$|v - v'| = u. \text{ Fix } v' .$$

Let $a = R/D$ (assumed $<< 1$).

We have two cases:

(i) $\underline{v \in [0, v']}$

Then

$$v' - v = u \text{ ; hence (3.5) yields } \frac{D}{R} < \frac{Q + v}{u} = \frac{Q + v'}{u} - 1 .$$

(3.7) $$u < [R/(D+R)] \cdot (Q+v') = \underline{u} .$$

Let

$$y = a\left(\frac{Q + v'}{u} - 1\right),\ u(y) = \frac{(Q + v')a}{y + a} \tag{3.8}$$

Then $u \in [0, \underline{u}]$ corresponds to $y \in [\infty, 1]$.

(ii) $\underline{v \in [v', v_0]}$

Then

$$v - v' = u;\ \frac{D}{R} < \frac{Q + v}{u} = \frac{Q + v'}{u} + 1$$

$$u < [R/(D-R)]\ (Q+v') = \bar{u}\ . \tag{3.9}$$

Let

$$z = a\left(\frac{Q+v'}{u} + 1\right);\ u(z) = \frac{(Q + v')a}{z - a} \tag{3.10}$$

$u \in [0, \bar{u}]$ corresponds to $z \in [\infty, 1]$.

For both cases,

$$dy = dz = -a\,\frac{(Q + v')}{u^2}\,du\ . \tag{3.11}$$

Also note the relations

$$du = \begin{cases} -dv & \text{case (i)} \\ +dv & \text{case (ii)} \end{cases} \tag{3.12}$$

nd

(.13) (i) $dy = a\,\dfrac{Q + v'}{u^2}\,dv$

(ii) $dz = a\,\dfrac{Q + v'}{u^2}\,dv$.

4. Probability density function, $g^*(v)$, for mix of P's strategies.

Let P choose strategy S_v (circle in speed-space radius v) with the following p.d.f.

$$g^*(v), \; 0 \leq v \leq v_o \quad \text{where} \int_{v=0}^{v_o} g^*(v) = 1 \; . \tag{4.1}$$

Given v', $0 \leq v' \leq v_o$ what is the expected 'coverage' (in the sense of the previous section) of the arc with radius v' ? It is

$$K(v') = \int_{v=v'-\underline{u}}^{v=v'+\bar{u}} \frac{v'}{v} \; (q^2 - v^2)^{1/2} \log \left\{ \frac{R/|v-v'|}{D/(q+v)} \right\} g^*(v) \, dv \; . \tag{4.2}$$

What is

$$H(v') = \Pr \{P \text{ kills } E \mid P\text{'s strategy} = g^*(v) \; ,$$

$$E\text{'s strategy} = v') \; ?$$

It is

$$H(v') = \min \{K(v') / 2\pi v', 1\} \; .$$

We make the assumption that it is sufficiently accurate to take

$$K(v')/2\pi v' \leq 1, \; v' \in [0, v_o] \; .$$

Then, referring to "A Smeared-Diagonal Zero-Sum Game", Section 10 of this paper we see that P's optimal strategy will yield

$$H(v') = \text{constant}, \; v' \in [0, v_o] ;$$

i.e.,

$$K(v')/v' = \text{constant} .$$

We seek the $g^*(v)$ that satisfies this constraint. It is heuristically apparent that E's optimal strategy will also be to choose v' using the same distribution $g^*(v)$. [See note on a smeared diagonal zero-sum game below] .

Without loss of generality, take $v_o = 1$ from now on. We write $g^*(v)$ in the following form:

$$(4.3) \qquad g^*(v) = \{a_o v + a_1 v^2 + a_2 v^3 + a_3 v^4\} (q^2 - v^2)^{-1/2}$$

$$= g(v) \cdot v \cdot (q^2 - v^2)^{-1/2} .$$

This imposes some restriction on $g(v)$, of course, but this form will be sufficiently general for practical purposes.

Note that, since

$$\int_o^1 g^*(v)\, dv = 1 ,$$

we should have

$$(4.4) \qquad a_o I_1 + a_1 I_2 + a_2 I_3 + a_3 I_4 = 1$$

where

$$I_i = \int_o^1 v^i (q^2 - v^2)^{-1/2} dv .$$

Using a certain criterion, we shall calculate the ratios

$$a_o : a_1 : a_2 : a_3 \; ,$$

Set $a_3 = -1$ arbitrarily, then compute

$$a_o I_1 + \ldots a_3 I_4 = C \; .$$

Then replace a_o, a_1, a_2, a_3 by a_o/C, a_1/C, a_2/C, a_3/C. Then, (4.4) will be satisfied.

Returning to (4.2) we have

$$\text{(4.5)} \qquad \frac{1}{v'} \; K(v') = \int_{v = v' - \underline{u}}^{v = v' + \bar{u}} \log \left\{ \frac{R / |v - v'|}{D / (Q + v)} \right\} g(v) \, dv$$

$$= \int_{v = v' - \underline{u}}^{v'} \text{---} \; dv + \int_{v = v'}^{v' + \bar{u}} \text{---} \; dv$$

$$= \int_{u = \underline{u}}^{0} \text{---} \; dv + \int_{u = 0}^{\bar{u}} \text{---} \; dv$$

$$= -\frac{1}{a(Q + v')} \left[\int_{y=1}^{\infty} --- u^2 \, (dy) + \int_{\infty}^{1} --- u^2 \, (-dz) \right]$$

using (3.8), (3.10), (3.13), and (3.11)

$$= -\frac{1}{a(Q + v')} \left[\int_{y=\infty}^{1} --- u^2 \, . \, dy + \int_{y=\infty}^{1} --- u^2 \, . \, dz \right]$$

$$= -\frac{1}{a(Q + v')} \left[\int_{y=\infty}^{1} \log y \, g(v' - u) \, . \, u^2 \, . \, dy \, [\text{case (i)}] \right.$$

$$\left. + \int_{z=\infty}^{1} \log z \, g(v' + u) \, . \, u^2 \, dz \right] [\text{case (ii)}]$$

Now, in case (ii), since

$$g(v) = a_o + a_1 v + a_2 v^2 + a_3 v^3$$

we have $g(v' + u) = a_o + a_1 (v' + u) + a_2 (v' + u)^2 + a_3 (v' + u)^3$

(drop the (')) $= (a_o + a_1 v + a_2 v^2 + a_3 v^3) \;\Big|\; = F_o$

$+ u (a_1 + 2 a_2 v + 3 a_3 v^2) \;\Big|\; + u \, . \, F_1$

$+ u^2 (a_2 + 3 a_3 v) \;\Big|\; + u^2 \, . \, F_2$

$+ u^3 (a_3) \;\Big|\; + u^3 F_3 \, .$

Let $F(u) = F_o + u F_1 + u^2 F_2 + u^3 F_3$

For case (i), we get $F(-u)$.

Now for $m \geq 2$, $a > -1$,

$$(4.6) \qquad I(m) = \int_{\infty}^{1} \frac{\log x}{(x+a)^m}\, du = \Big[-\frac{\log x}{(m-1)(x+a)^{m-1}}$$

$$+ \frac{1}{(m-1)a} \Big\{ \frac{1}{(m-2)(x+a)^{m-2}} + \frac{1}{(m-3)\,a\,(x+a)^{m-3}} + \frac{1}{2a^{m-4}(x+a)^2}$$

$$+ \frac{1}{a^{m-3}(x+a)} \Big\} + \frac{1}{(m-1)a^{m-1}} \log\left(\frac{x}{x+a}\right) \Big]_{\infty}^{1}$$

$$= + \frac{1}{(m-1)a} \Big\{ \frac{1}{(m-2)(1+a)^{m-2}} + \ldots \frac{1}{a^{m-3}(1+a)} - \frac{1}{a^{m-2}} \log(1+a) \Big\}$$

and, for $a \ll 1$, we get, using,

$$\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} - - ,$$

$$(4.7) \quad I(2) = -\frac{1}{a}\log(1+a) = -\left(1 - \frac{a}{2} + \frac{a^2}{3} - \frac{a^3}{4} + \frac{a^4}{5}\right)$$

$$I(3) = \frac{1}{2a}\left[\frac{1}{1+a} - \frac{1}{a}\log(1+a)\right]$$

$$\doteq \frac{1}{2a}\left(-\frac{1}{2}a + \frac{2}{3}a^2 - \frac{3}{4}a^3\right) = \frac{1}{a}\left(-\frac{1}{4}a + \frac{1}{3}a^2 - \frac{3}{8}a^3\right)$$

$$I(4) = \frac{1}{3a}\left[\frac{1}{2(1+a)^2} + \frac{1}{a(1+a)} - \frac{1}{a^2}\log(1+a)\right]$$

$$\doteq \frac{1}{a^2}\left(-\frac{1}{9}a^2 + \frac{1}{4}a^3\right)$$

$$I(5) = \frac{1}{4a}\left[\frac{1}{3(1+a)^3} + \frac{1}{2a(1+a)^2} + \frac{1}{a^2(1+a)} - \frac{1}{a^3}\log(1+a)\right]$$

$$\doteqdot \frac{1}{a^3}\left(-\frac{1}{16}a^3\right)$$

let $\quad I^-(2) = I(2)$ with a replaced by $-a$, $I^-(3) = \text{---}$ etc.

Thus, in (ii)

$$(4.8) \qquad \int_{z=\infty}^{1} \log z . g(v+u)\, u^2 . dz$$

(writing now v for v') make the substitution

$$(4.9) \qquad u = \frac{a(Q+v)}{z-a}$$

to obtain

$$(4.10) \qquad -\int_{z=\infty}^{1} \log z . (F_o + F_1 . u + F_2 . u^2 + F_3 u^3) u^2\, dz$$

$$= -\int_{z=\infty}^{1} \log z \left\{ F_o \left[\frac{a(Q+v')}{z-a}\right]^2 + F_1 \left[\frac{a(Q+v')}{z-a}\right]^3 + \dots \right\} d$$

$$= F_o . I^-(2) . a^2 (Q+v)^2 + F_1 I^-(3)\, a^3 (Q+v)^3$$

$$+ F_2 I^-(4)\, a^4 (Q+v)^4 + F_3 I^-(5)\, a^5 (Q+v)^5$$

Similarly

$$(4.11) \qquad \int_{y=\infty}^{1} \log \{ F_o [\frac{a(Q+v)}{z+a}]^2 - F_1 \ldots \} \, dy = F_o \, I(2)\, a^2 (Q+v)^2$$

$$- F_1 \, I(3)\, a^3 (Q+v)^3$$

$$+ F_2 \, I(4)\, a^4 (Q+v)^4$$

$$- F_3 \, I(5)\, a^5 (Q+v)^5 .$$

Hence, from (4.5),

$$(4.12) \qquad K(v) = - \frac{1}{a(Q+v)} [\int -- + \int --]$$

$$= - \frac{1}{a(Q+v)} \quad [F_o \cdot a^2 (Q+v)^2 \cdot \{ I(2) + I^-(2) \}$$

$$+ F_1 \cdot a^3 (Q+v)^3 \cdot \{ - I(3) + I^-(3) \}$$

$$+ F_2 \cdot a^4 (Q+v)^4 \cdot \{ I(4) + I\text{-}(4) \}$$

$$+ F_3 \cdot a^5 (Q+v)^5 \cdot \{ - I(5) + I^-(5) \}]$$

$$= - \frac{1}{a(Q+v)} \cdot a^2 (Q+v)^2 \Big[F_o [- 2 - \frac{2}{3} a^2 - \frac{2}{5} a^4]$$

$$+ F_1 (Q+v) [\frac{1}{2a} + \frac{3}{4} a^3]$$

$$+ F_2 (Q+v)^2 [- \frac{2}{9} a^2]$$

$$+ F_3 (Q+v)^3 [\quad \frac{1}{8} a^3]$$

Therefore

(4.13) $$\frac{1}{av} \cdot K(v) = (Q + v) \{ F_o \left(2 + \frac{2}{3} a^2 + \frac{2}{5} a^4\right)$$

$$+ (Q+v) F_1 \left(- 1/2a - 3/4\, a^2\right)$$

$$+ (Q + v)^2 F_2 \left(\frac{2}{9} a^2\right) + (Q + v)^3 F_3 \left(- \frac{1}{8} a^3\right) \}$$

$$= P \cdot (Q + v) (a_o + a_1 v + a_2 v^2 + a_3 v^3)$$

$$+ S \cdot (Q + v)^2 (a_1 + 2 a_1 v + 3 a_3 v^2)$$

$$+ T \cdot (Q + v)^3 (a_2 + 3 a_3 v)$$

$$+ U \cdot (Q + v)^4 (a_3)$$

Where

(4.14) $$P = \left(2 + \frac{2}{3} a^2 + \frac{2}{5} a^4\right),\ S = \left(- \frac{1}{2} a - \frac{3}{4} a^3\right),\ T = \frac{2}{9} a^2,\ U = - \frac{1}{8} a^3$$

$$= \{ a_o \cdot Q \cdot P + a_1 \cdot Q^2 \cdot S + a_2 \cdot Q^3\, T + a_3\, Q^4\, U \}$$

$$+ v \{a_o P + a_1 Q (P + 2 S) + a_2 Q^2 (2 S + 3 T) + a_3 Q^3 (3T + 4U)\}$$

$$+ v^2 \{a_1 (P + S) + a_2 Q (P + 2S + 3T) + a_3 Q^2 (3S + 9T + 6U)\}$$

$$+ v^3 \{a_2 (P + 2S + T) + a_3 Q (P + 6S + 9T + 4U)\}$$

$$+ v^4 \{a_3 (3T + U)\}$$

We wish $\frac{1}{a_v} K(v)$ to be approximately constant. So we take

(4.15) $$a_3 = -1 .$$

$$a_2 = - a_3 \cdot Q \cdot \frac{(P + 6S + 9T + 4U)}{P + 2S + T}$$

$$a_1 = \frac{- a_2 \cdot Q (P + 2S + 3T) + a_3 \cdot Q \cdot (3S + 9T + 6U)}{P + S}$$

$$a_o = - \frac{a_1 (Q + 2S) + a_2 \cdot Q^2 (2S + 3T) + a_3 Q^3 (3T + 4U)}{P}$$

(The choice of $a_3 = - 1$ is arbitrary).

5. Aids to Calculation of the Normalizing Factor for g*(v)

The values of the integrals of (4.4),

$$I_i = \int_0^1 v^i \, (q^2 - v^2)^{-1/2} \, dv$$

are as follows:

(5.1) $$I_1 = \frac{1}{3} \left[q^3 - (q^2 - 1)^{3/2} \right]$$

(5.2) $$I_2 = \frac{1}{8} \left[(2 - q^2)\,(q^2 - 1)^{1/2} + q^4 \sin(1/q) \right]$$

(5.3) $$I_3 = \frac{1}{15} \left[-(3+2q^2)\,(q^2 - 1)^{3/2} + 2q^5 \right]$$

(5.4) $$I_4 = \frac{1}{48} \left[(8 - 2q^2 - 3q^4)\,(q^2 - 1)^{1/2} + 3q^6 \sin^{-1}(1/q) \right]$$

6. Numerical example.

Take $Q = 2.0$

$q = 2.0$

$R = .05$

$D = 2.0$

The 'recommended' values for a_0, a_1, a_2, a_3 using (4.15) are as follows (with a_3 taken arbitrarily as -1.0)

(6.1) $a_0 = 7.796$

$a_1 = -3.901$

$a_2 = 1.902$

$a_3 = -1.000$

On substituting $q = 2.0$ in equations (5.1) - (5.4), we obtain

$I_1 = .934$

$I_2 = .614$

$I_3 = .456$

$I_4 = .362$

Then $C = a_0 I_1 + \ldots a_3 I_4 = 5.392$, $C^{-1} = 0.1854$.

Therefore, the recommended strategy for P is given by

$$g^*(v) = C^{-1} (a_0 v + a_1 v^2 + a_2 v^3 + a_3 v^4) (q^2 - v^2)^{-1/2}$$

We tabulate $g^*(v)$ at intervals of $v = .1$, in table 1.

(The columns headed $g^{**}(v)$, $g^*_{J\,P}(v)$ are discussed later.)

As a check, note that if the numerical integration formula

$$(1/2) \sum_{i=0}^{9} (0.1). \left\{ g^* \left[i(0.1) \right] + g^* \left[(i+1)(0.1) \right] \right\}$$

is used, we obtain $\int g^*(v)\, dv = 0.990$ approximately.

(1) v	(2) $g^*(v)$	(3) $g^{**}(v)$	(4) $g^*_{LP}(v)$
0.0	0.000	.000	0.000
0.1	0.275	.262	0.261
0.2	0.449	.4[illegible]	0.480
0.3	0.744	.7[illegible]	0.644
0.4	0.941	.[illegible]	0.760
0.5	1.112	1.0[illegible]	0.853
0.6	1.257	1.19[illegible]	0.965
0.7	1.374	1.311	1.153
0.8	1.462	1.382	1.485
0.9	1.518	1.435	2.036
0.95	(1.529)	(1.931)	(2.459)
1.0	1.541	2.914	2.882

Table I - Recommended Strategies for the Pursuer, P.

(Note: For $v = 0.95$, interpolated values have been calculated.)

(Note: For $v' = 1$, $\underset{\sim}{u} = \frac{R}{D+R}(Q+v') = \frac{0.05}{2.0+0.05}(2.0+1.0) \doteq .075$).

Then the following 'Kill probabilities' are obtained (Col. 3 is discussed below)

(1) If E adopts speed v' with random azimuth	(2) 2π x P's Kill Probability, using strategy $g^*(v)$	(3) 2π x P's Kill Probability, using strategy $g^*_{LP}(v)$
.025	.232	.221
.05	.268	.255
.075	.284	.271
.1	.289	.274
.2	.289	.265
.3	.289	.250
.4	.288	.233
.5	.287	.221
.6	.286	.221
.7	.283	.241
.8	.279	.289
.9	.265	.358
1.0	.133	.221

Table 2 - Kill Probabilities

Thus, when strategy $g^*(v)$ is used, the approximate relation

$$P(\text{Kill} \quad \text{E uses strategy } v') = \text{constant}$$

is demonstrated, except for v' in the region of 1.0, which is discussed below. (Note: In the numerical integration used for this table, an interval of 0.001 was employed).

The kill probability for the $g^{**}(v)$ case was not calculated explicitly (see section 7 below), but a rough calculation given below shows a typical kill probability of 0.267.

7. Correction to $g^*(v)$ at the rim of the Speed Circle

The approximation used in Section 4, assumes that $g^*(v)$ is defined in $(0,\infty)$. In fact, $g^*(v)$ is truncated at $v = v_o$ (=1, by choice of units).

Consider, for E, the maximum speed strategy $v' = v_o$. Then, in eq. (4.5) we assumed

$$\frac{1}{v'} K(v') = \int_{v = v' - \underline{u}}^{v = v' + \bar{u}} H(|v - v'|)\, v^{-1} g^*(v)\, dv \tag{7.1}$$

where $K(v')$ was the "coverage" (in the sense of Section 3, of the arc v', and $H(|v - v'|)$ is a function of v which is approximately symmetric about $v = v'$.

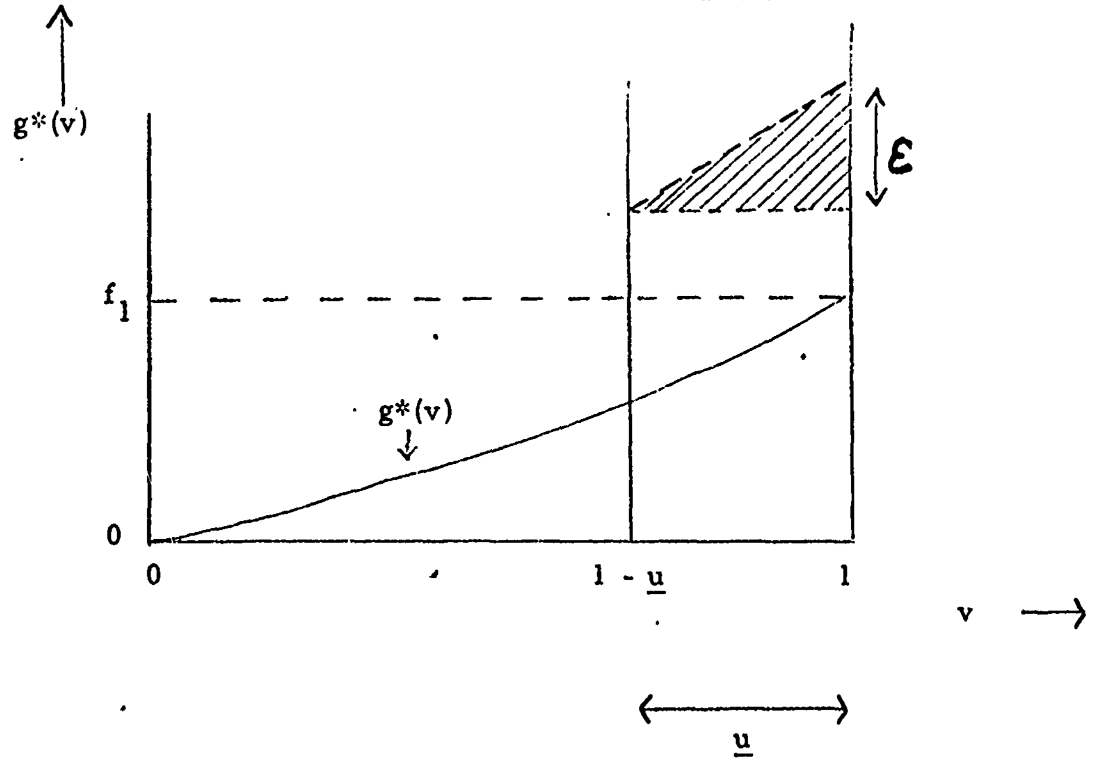
In reality, we have

$$\frac{1}{v'} K(v') = \int_{v = v' - \underline{u}}^{0} H(|v - v'|) v^{-1} g^*(v)\, dv \tag{7.2}$$

Assume that $v^{-1} g^*(v)$ is approximately constant within the above limits of integration, then

$$\int_{v' = v' - \underline{u}}^{0} \sim (1/2) \int_{v' - v' - \underline{u}}^{v' = v' + \bar{u}} \tag{7.3}$$

We chose $g^*(v)$ so that $K(v')/v'$ would be approximately constant, but from (7.3) it is clear that $K(v')/v'$ will drop to one half of this constant value, for v' in the neighborhood of $v' = 1$.



Now $\underline{u} = \frac{R}{D+R}$ (Q + v'); let $g^*(1) = f_1$, $\epsilon = f_1/(1 + \underline{u}\, f_1/2)$ and note that $f_1(1 - \underline{u}\epsilon/2) = \epsilon$.

We attempt to correct the error referred to above, by adding a "wedge" of probability mass to g*(v) in the range (1 - $\underline{u}$, 1). Let us, therefore, change g*(v) to g**(v), where

$$g^{**}(v) = \begin{cases} g^*(v) \cdot (1 - \underline{u}\ \epsilon/2) & 0 \leq v \leq 1 - \underline{u} \\ g^*(v)\ (1 - \underline{u}\ \epsilon/2) + \epsilon \dfrac{v' - (1 - \underline{u})}{\underline{u}} & , 1 - \underline{u} < v \leq 1 \end{cases}$$

Note that

$$\int_0^1 g^{**}(v)\, dv = (1 - \underline{u}\epsilon/2) \int_0^1 g^*(v)\, dv + \underline{u}\epsilon/2 = 1.$$

Also $\quad g^{**}(1) = f_1(1 - \underline{u}\epsilon/2) + \epsilon = 2\epsilon$

Relatively speaking, our transformed g**(v) has doubled the weight assigned to v' = 1, and this extra weight tapers off (linearly) to zero at v' = 1 - $\underline{u}$.

For the numerical example, we give the transformed g**(v) in Table I, col. (3).

It is seen that $g^{**}(v)$ is quite close to the strategy $g^*_{LP}(v)$, discussed below. Since kill probabilities are calculated for $g^*_{LP}(v)$, they can be expected to be very similar for $g^{**}(v)$, and hence were not calculated for this latter case explicitly. However, a rough idea is obtained by applying the correcting factor, 0.9454 in this case, to a 'typical' kill probability in the middle range for v in column (2) of Table 2, say the 0.283 value. We obtain 0.267; this compares very favorably with the minimum value of 0.221 obtained with the LP solution, restricted to a mixture of four pure strategies. Of course, if more than four strategies were mixed, the LP value would rise. Also, we have not checked that our 'ad-hoc' solution surpasses the 0.267 at the two end-points, $v = 0, 1$.

8. <u>Solution of the Game by Linear Programming Methods.</u>

Suppose we write P's strategy in the form given in (4.3), namely

$$g^*(v) = g(v) \cdot v \cdot (q^2 - v^2)^{-1/2}$$

and, now, let

$$g(v) = a_o + a_1 v + a_2 v^2 + \dots + a_n v^n.$$

We may consider each term,

$$a_i v^i$$

as corresponding to a 'pure strategy'; that is, given the p.d.f.

$$g(v) = a_i v^i$$

(suitably normalized), P samples from this distribution to obtain his strategy v.

For this pure strategy, P_i say, we can compute

$$E_{(v \,\tilde{d}\, a_i v^i)} \Pr(\text{Kill} \mid P \text{ adopts } v \text{ ; } E \text{ adopts } v')$$

where $\tilde{d}$ denotes "is distributed as"

These quantities, for i = 1, . . n, become the row entries in the following matrix zero-sum game:

P's strategies

$$\begin{array}{cc} & \begin{array}{cccc} a_o & "a_i v" & \dots & "a_n v^n" \end{array} \\ \begin{array}{c} v' = E_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ E_{13} \end{array} & \boxed{\begin{array}{cccc} & & & \\ \\ \\ \\ \\ \\ \\ \end{array}} \end{array}$$

E's strategies

The solution of this game yields the optimal strategy for P, and the optimal strategy for E which will consist of allocation of effort to the points $E_1 \ldots\ldots E_{13}$ say.

Note that E's strategy will not in general be to allocate effort proportional to v' .

In fact, consider the numerical example used previously. The linear programming approach, used (for simplicity) with a g(v) of the form

$$a_o + a_1 v + a_2 v^2 + a_3 v^3$$

and with strategies

$$.1, .2, \ldots .1.0$$

for E, yields the following solution:

$$\text{P's strategy: } \left.\begin{aligned} a_o &= 1.3691 \\ a_1 &= -.3192 \\ a_2 &= -3.2451 \\ a_3 &= +3.8591 \end{aligned}\right\} g^*_{LP}(v), \quad \text{say}$$

E's strategy:	v' =	.025,	Pr	.0774
		.5	Pr	.0991
		.6	Pr	.4263
		1.0	Pr	.3972
		Total		1.0000

This strategy for P results in the kill probabilities shown in column (3) of Table 2.

The underlined entries in this column indicate how E's strategies concentrate on P's 'weak spots'. In fact, since P's strategy, because of the assumption on g(v), results in approximately Pr (kill) of 4th degree polynomial form, it is sufficient for E to concentrate his efforts on four points: one at the boundary v = 1 where P (Kill) is decreasing with v (because of the edge effect at the rim of the speed circle), two adjacent points at the interior minimum which occurs between v = .5, v = .6, and one at the center, v = .025.

9.0 Effect of time constraints on E's strategies.

(9.1) Let $k(v) = \Pr(\text{Kill} \mid E$ adopts speed v, P adopts a strategy drawn from $g^*(v))$.

Suppose that there is a good reason for E, in addition to avoiding detection, to get as far away as possible from O, in a stated fixed time interval.

For example, it might be known (to both players) that additional P forces can arrive on the scene in 1 hour's time.

To pin down this factor, let us define

(9.2) $k_2(v) = \Pr(\text{Kill} \mid E$ adopts speed v, no kill has been made by time $t=1$, P adopts some optimal strategy using all his forces after time $t=1)$

and represent $k_2(v)$ by

(9.3) $$k_2(v) = \lambda\,(1-v^2)\ .$$

Then the total payoff to P is

(9.4) $$\begin{aligned} k_3(v) &= \Pr(\text{Kill by } t=1) + \Pr(\text{Kill after } t=1) \\ &= k(v) + [1-k(v)]\,k_2(v) \\ &= k(v) + [1-k(v)]\,(1-v^2)\,\lambda\ . \end{aligned}$$

As in section 4, we require for an optimal solution, that

$$(9.5) \qquad k_3(v) = \text{constant, independent of } v, = k \text{ say .}$$

Now, an approximate expression for $k(v)$ is available in eq. (4.13). We take the variable $K(v)$ given there, and multiply by the correction factor C^{-1} as specified in eq. (4.4) et seq., to obtain $k(v)$.

This expression for $k(v)$ may be substituted into (9.4), and then we may choose the constants a_o, a_1, a_2, a_3, (a_3 chosen arbitrarily to be -1), so as to satisfy (9.5) to third order in v .

10. Note on a Smeared-Diagonal Zero-Sum Game

10.1 Purpose

The purpose of this note is to provide analytical support for an intuitively obvious fact.

We have two players I and II, and certain physical regions $R_1 \ldots R_n$ in which they can allocate 'effort' (e.g., for a search game, the 'effort' will be the percentage of time they are in region R_i). The amount of effort will be described by q_i (Player II), and p_j (Player I), and the pay-off will be

$$\sum_{i,j=1,\ldots n} q_i C_{ij} p_j$$

to player II, where the C_{ij} are (almost) zero unless $i=j$. Our heuristic reasoning is as follows:

Player II adjusts his q_i so that the quantity

$$\sum_{i=1,\ldots n} q_i C_{ij}$$

is approximately constant (independent of j). Then it is claimed (i) that this is II's approximate optimal strategy, and (ii) that the strategy for I which consists of choosing the same vector for the p's that II chose for the q's, will be approximately optimal.

This claim (i) is justified for the model o· the paper as follows:

First, by the Min-max theorem, there exists solutions $\{q_i\}$, $\{p_j\}$ and a constant v (the value of the game such that

$$\sum_{i,\, j = 1, \ldots n} q_i \; C_{ij} \; p_j = v$$

Now suppose $\sum_i q_i \; C_{ij} > v$ for some j ...

Then no solution, for I, $\{p_j\}$, can ... j ...

However, we may assume from the ... game, that it would not be reasonable for any particular speed ... to be unavailable to the Evader as part of a solution. We conclude that

any solution for II must have $\sum_i q_i \; C_{ij}$ = constant, independent of j.

10.2 Application

Apply this to a search game performed over the unit circle, divided into concentric annuli of equal width. The unit circle could be in speed space, or in real space but in any case we require that the sub-strategy for I and II consisting of picking a particular annulus shall result in a uniform distribution over the annulus.

A pay-off of unity ("success") occurs, to II, if the two players come within some (small) distance of each other.

Let us suppose that II allocates a small amount of effort - say 10% -to a particular annulus. Assume that I has concentrated all his effort on

that annulus. Then the probability of "success" is proportional to 1/(Area of annulus). Thus II allocates his effort in proportion to the area of the annulus, that is, linearly increasing with the radius of the annulus. Similarly, I allocates his effort in the same proportions.

10.3 Detailed Analysis

Consider the following zero-sum game

Player II, P

		q_1	q_{i-1}	q_i	q_{i+1}	......
	p_1					
	p_{i-1}		$\frac{1}{A_{i-1}}$	$\frac{\epsilon(i)}{A_i}$		
Player I, E	p_i		$\frac{\epsilon(i-1)}{A_{i-1}}$	$\frac{1}{A_i}$	$\frac{\epsilon(i+1)}{A_{i+1}}$	
	p_{i+1}			$\frac{\epsilon(i)}{A_i}$	$\frac{1}{A_{i+1}}$	
	.				-----	
	.				-----	

The constants $\epsilon(i)$ are supposed small compared to 1. Also the $\epsilon(i)$ are supposed to vary slowly with i, and so are the A_i. Consider mixed strategies for I, II specified by $\{p_i\}$, $\{q_i\}$. Then the payoff (from II's point of view) will be

$$\sum_{i=1}^{n} p_i \underbrace{\left\{\frac{\epsilon(i-1)}{A_{i-1}} q_{i-1} + \frac{1}{A_i} q_i + \frac{\epsilon(i+1)}{A_{i+1}} q_{i+1}\right\}}_{R_i,\ \text{say}}$$

Suppose we can find a strategy, $\{q_i^*\}$, say, such that the R_i are approximately constant (independent of i). Then we assert (i) $\{q_i^*\}$ is approximately the optimal strategy for II, (ii) the strategy $\{p_i^*\} = \{q_i^*\}$ is approximately optimal for I. (i) is obvious, since then the payoff of the game does not depend on I's strategy, which is the condition we have assumed for optimality. (ii) is seen as follows:

The pay-off with the suggested strategy $\{p_i^*\} = \{q_i^*\}$ is

$$\sum_{i=1}^{n} q_i \left\{\underbrace{p_{i-1}^* \frac{\epsilon(i)}{A_i} + p_i^* \frac{1}{A_i} + p_{i+1}^* \frac{\epsilon(i)}{A_i}}_{S_i,\ \text{say}}\right\}$$

But $$S_i = \left\{q_{i-1}^* \frac{\epsilon(i)}{A_i} + q_i^* \frac{1}{A_i} + q_{i+1}^* \frac{\epsilon(i)}{A_i}\right\}$$

and we have, since R_i = const. for $q_i = q_i^*$,

$$\left\{ q_{i-1}^* \frac{\epsilon(i-1)}{A_{i-1}} + q_i^* \frac{1}{A_i} + q_{i+1}^* \frac{\epsilon(i+1)}{A_{i+1}} \right\} = \text{const. (indep. of } i)$$

Comparison of these last two expressions shows that S_i is approximately independent of i , and thus the suggested strategy is approximately optimal.

11. The Two-Stage Speed Game

11.0 Description of Model

The following is a suggested approach to solving the two-stage (for E) speed game. No analytical results are obtained.

The same initial situation is assumed. At time $t=0$, an Evader (E) is sighted at a point O.

He realizes he has been sighted, dives, and adopts the following two-stage strategy:

(i) E shooses a direction at random,

(ii) E adopts a fixed speed w, $0 \leq w \leq v_o$,

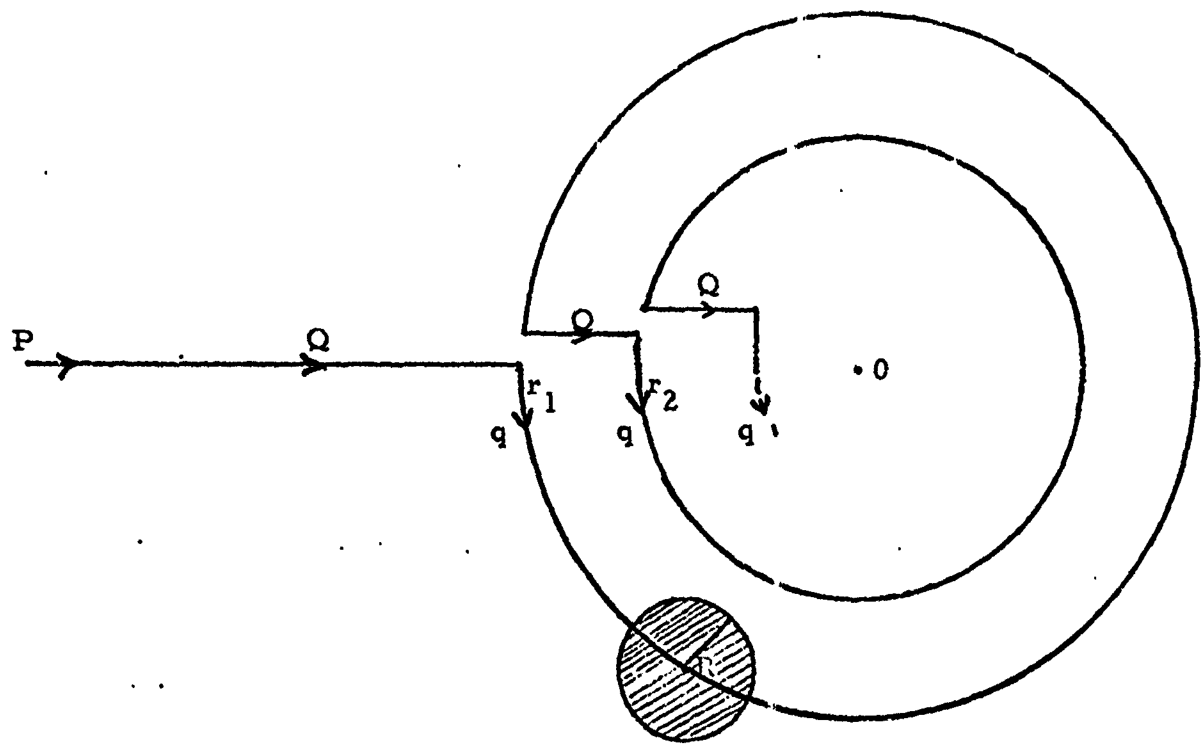
until time T, then adopts a fixed speed v, $0 \leq v \leq v_o$ thereafter.

A Pursuer (P) is located at distance D from O; we suppose that E knows P's maximum speed and distance d, but not his bearing.

Previously we considered strategies for P which consisted of circles in the 'speed-space' of E, corresponding to spirals of constant pitch in real space. This choice is no longer convenient.

Instead we shall consider the following family $\mathcal{F}_1$ say, of strategies for P. P steams at full speed (Q) towards O, and stops short at distance r_1 (> 0) from O. At speed q, he then performs search around a circle radius r_1, center O. Having completed the circle, he proceeds along the radius to a point distance r_2, and performs another circle in the same sense as the first, and so on.

The trips between successive starting points of the circles are made at speed Q, and the circles themselves with speed q.



The same "cookie-cutter" radius of detection of E by P will be assumed; however the radii of the cookie-cutter R_1, R_2 will be dependent on the respective speeds w, v of E, according to some noise-law to be specified.

It is claimed that the family $\mathfrak{F}_1$ is "almost-dense" or "almost-good" in the sense that the results achieved by P through a suitable mix of $\mathfrak{F}_1$ will be almost as good as those that could be obtained with any general strategy. The objects of the game are, for E, escape without detection, and for P, detection of E.

It is conjectured that the optimal strategy for E will be the following:

(i) Adopt the first speed w up to time T, such that the probability of detection by T (based on an a priori knowledge of the initial distance-to-go D) is reasonably small;

(ii) Adopt the second speed v by sampling from a p.d.f. in such a way that the distribution : fixed time t of the position of E is approximately uniform over some circle , or some area formed by subtracting from the largest circle $v_o.t$, an inner circle of some radius.

Since the speed v is fixed, once chosen, it is not claimed that this distribution can necessarily be obtained for all values of t . However, it is expected that the true distribution will be approximately uniform, for each of a wide range of values of t .

11.1 Method of Analysis

Regarding the initial choice of radius r_1 by the Pursuer. We may assume that r_1 is chosen by drawing from a distribution g(r) – on analogy with the p.d.f. g(v) of the first model.

Regarding the second and subsequent choices of radius r , we may suppose these are made in such a way as to replicate the distribution g(r) . We shall ignore the effect of the time taken to move between successive circles radii r_1 , r_2 .... .

From now on, we may concentrate on the distribution g(r) .

11.2 Choice of Evader Strategies

It will be necessary to restrict our study to a finite number of pure strategies. These will be written

(w,v,T)

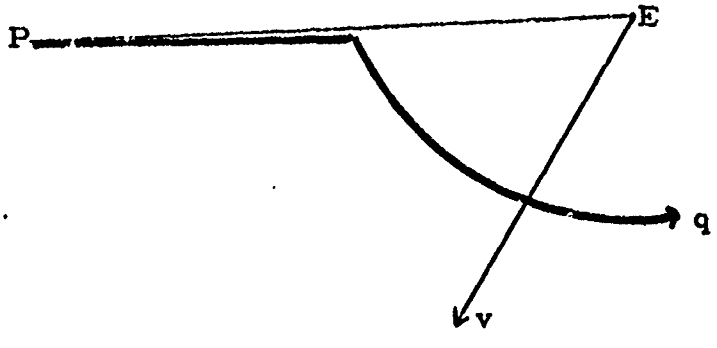
corresponding to the strategy

$$(11.1)\qquad \begin{cases} \text{speed } w, & 0 \le t \le T \\ \text{speed } v, & t > T \end{cases} .$$

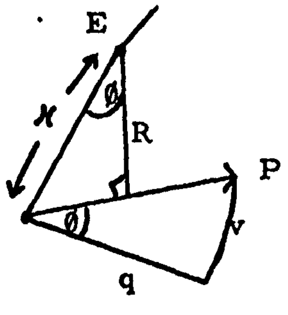
For each of these pure strategies, we compute the probability of kill, thus

$$\text{(11.2)} \qquad P(\text{Kill} \mid (w,v,T);\ g(r)) = \int_{r=0}^{\infty} P(\text{Kill} \mid (w,v,T);\ P\text{'s strat.} = r)\ .\ dr$$

First, note the usual correction for the point of closest approach of two bodies in (approximate) straight-line motion.



Let E's speed be generally v . P's motion relative to E is then:



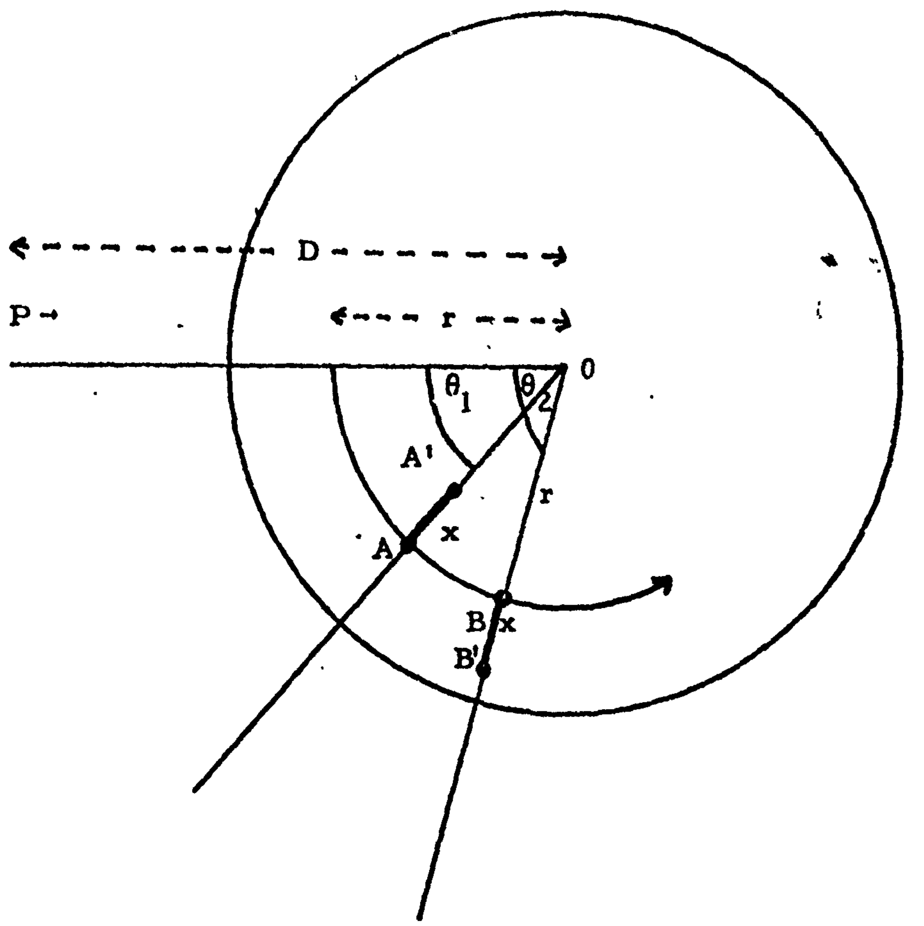
$$\frac{R}{x} = \cos \phi = \frac{q}{(v^2 + q^2)^{1/2}} = \frac{1}{(1 + \frac{v^2}{q^2})^{1/2}}$$

Therefore

$$x = R\ (1 + \frac{v^2}{q^2})^{1/2}$$

Hence "kill" occurs if, at the moment P crosses the path of E , the distance between them is less than

$$\text{(11.3)} \qquad x = R(1 + v^2/q^2)^{1/2} \ .$$



(Case 1) Suppose 'kill' occurs during the 'w' of first phase of the (w,v,T) policy. We shall derive the range of the argument, (θ_1, θ_2) such that Kill results if E should choose θ in this range, as follows: Suppose that 'Kill' occurs on the first circumnavigation at radius r, by P. The value of θ_1 is given by the following consideration.

P must be at point A when E is at point A', precisely distance x closer to 0.

Thus, provided $r < wT + x$,

(11.4) θ_1 is given by

$$\left[\frac{D-r}{Q} + \frac{\theta_1 . r}{q}\right] = \frac{r-x}{w}$$

Similarly, provided $r > wT - x$,

(11.5) θ_2 is given by

$$\left[\frac{D-r}{Q} + \frac{\theta_2 . r}{q}\right] = \frac{r+x}{w}$$

(Case 2)
Suppose "Kill" occurs during the "v" part of the (w,v,T) strategy. Then the corresponding equations are:

(11.6) $$\theta_1 \Leftarrow \left[\frac{D-r}{Q} + \frac{\theta_1 . r}{q}\right] = T + \frac{(r-x) - Tw}{v}$$

(11.7) $$\theta_2 \Leftarrow \left[\frac{D-r}{Q} + \frac{\theta_2 . r}{q}\right] = T + \frac{(r+x) - Tw}{v}$$

Note that, conceivably, solutions in excess of 2π may arise (solutions for θ which are negative will be rejected, that is the corresponding range will start at $\theta = 0$).

Note also that, since the ranges for r for the two cases – w-kill and v-kill, overlap, the same path for E may result in Kill in both cases.

From these two considerations we see that we must be careful to avoid counting the same kill twice.

Thus, the ranges for θ must be added together on an 'or' basis as follows: The ranges must first be expressed in terms of coverage of the interval $[0, 2\pi]$. Then Range for θ such that kill results

= (Range such that w-kill results)

$\cup$ (Range such that v-kill results)

On dividing the result by 2π, we have precisely the probability defined in (II.2), integrand, namely

$$P(\text{Kill} \mid (w, v, T);\ P\text{'s strat.} = r)$$

On integrating with respect to the p.d.f. $g(r)$ we obtain

$$P(\text{Kill} \mid (w, v, T);\ P\text{'s strat.} = g(r))$$

It should then be possible to obtain approximate min-max strategies on each side, either by analytical means, or by a Linear Programming approach, as in the first Model.

The resulting value of the game (setting $T = \infty$) should be comparable with the value for the first game, if, indeed, the 'circular' strategies are about as 'good' as the spiral strategies considered in the first Model.

12. <u>An alternative approach to the two state speed game by adapting the one-stage speed game</u>.

12.0 To some extent, we may approximate the two-stage speed game by adapting the one-stage speed results previously given (Model I). We consider the same spiral strategies for P as were used in Model I).

The Evader (E) is restricted to the following sub-set of two-stage speed strategies, by modifying condition (ii) of section 11.0

(ii) E adopts maximum speed v_o for time λt_o, where

$$t_o = D/(Q + v_o) \;,\; O \leq \lambda \leq 1 \;,$$

then adopts fixed speed v, $O \leq v \leq v_o$, thereafter.

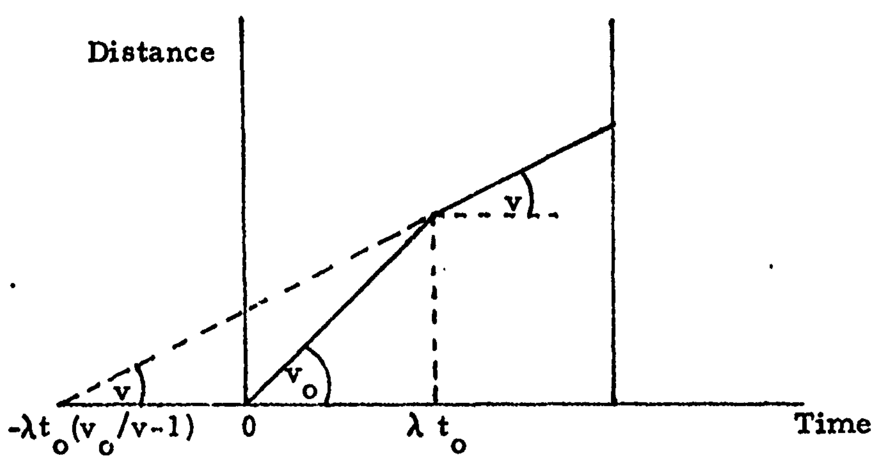
The constant λ is known to P .

Since the class of E-strategies (ii)' includes the class used in Model 1, the value of the game for E must be at least as good as that in Model 1.

12.1 Conclusions

An analytical form for P's optimal strategy is exhibited.

12.2 Technical Discussion



The E-strategy (v_o, v) is seen to be equivalent to a single speed strategy v , where the time-origin is backed-off $-\lambda t_o (v_o/v - 1)$.

[It will be noticed that this arrangement is not defined for $v = 0$; since the strategy $v = 0$ in practice falls outside the scope of our model, it will be assumed that both P and E assign zero weight to this strategy.]

The situation, given E chooses (v_o, v) , is thus equivalent to Model 1, with P starting, at $t = 0$, at a distance

$$D' = D + Q \cdot \lambda \; t_o \, (v_o/v-1)$$

$$= D + Q \cdot \lambda \left(\frac{v_o - v}{v}\right) \left(\frac{D}{Q + v_o}\right)$$

$$= D \left\{1 + \lambda \frac{v_o - v}{Q + v_o} \cdot \frac{Q}{v}\right\}$$

$$= D \left\{1 - \lambda \frac{Q}{Q + v_o}\right\} + D \cdot \lambda \frac{v_o}{Q + v_o} \cdot \frac{Q}{v}$$

$$= D^* \left\{1 + \mu \, v^{-1}\right\}$$

where

$$D^* = D \left\{1 - \lambda \, Q/(Q + v_o)\right\}$$

$$\mu = \lambda \, v_o \, Q/(Q + v_o - \lambda \, Q)$$

and will be assumed small.

Note that, with this fictional substitution for D , the analysis of Model 1 carries through to (and including) eq, (4.14), with the following exceptions:

(1) Equation (3.2) is good only to first order in v ;

(2) $a = R/D$ must be replaced by

$$a' = R/D^*(1 + \mu v^{-1})$$
$$\doteq a^* (1 - \mu v^{-1}) \text{ (first order in } \mu) .$$

where we now use symbol a^* for R/D^*. To first order in μ , we may write

$$a'^2 = a^2 (1 - 2 \mu v^{-1}) \text{ , etc.}$$

In the spirit of the approximations made so far, we consider the P-strategy (c.f., eq. (4.3))

$$g^*(v) = g(v) \cdot v(q^2 - v^2)^{-1/2}$$

where

$$g(v) = a_o + a_1 v .$$

Then, from eq. (4.13),

$$\frac{1}{a' v} k(v) = (2 \cdot \frac{2}{3} a'^2 + \frac{2}{5} a'^4) (Q + v) (a_o + a_1 v)$$

$$+ \left(- \frac{1}{2} a' - \frac{3}{4} a'^3 \right) (Q + v)^2 a_1$$

$$= \left[2 + \frac{2}{3} a^{*2} (1 - 2 \mu v^{-1}) + \frac{2}{5} a^{*4} (1 - 4 \mu v^{-1})\right] (Q + v) (a_o + a_1 v)$$

$$+ \left[- \frac{1}{2} a^* (1 - \mu v^{-1}) - \frac{3}{4} a^{*3} (1 - 3 \mu v^{-1})\right] (Q + v)^2 a_1$$

$$= \left[P + v^{-1} \left(- \frac{4}{3} a^{*2} - \frac{8}{5} a^{*4}\right) \mu\right] (Q + v) (a_o + a_1 v)$$

$$+ \left[S + v^{-1} \left(\frac{1}{2} a^* + \frac{9}{4} a^{*3}\right) \mu\right] (Q + v)^2 a_1$$

$$= (P + G v^{-1}) (Q + v) (a_o + a_1 v)$$

$$+ (S + H v^{-1}) (Q + v)^2 a_1$$

where $G = \left(- \frac{4}{3} a^{*2} - \frac{8}{5} a^{*4}\right) \mu$, $H = \left(\frac{1}{2} a^* + \frac{9}{4} a^{*3}\right) \mu$,

$P = 2 + \frac{2}{3} a^{*2} + \frac{2}{5} a^{*4}$, $S = - \frac{1}{2} a^* - \frac{3}{4} a^{*3}$.

Hence $\frac{1}{a^* v} k(v) \doteq (1 - \mu v^{-1})$.

$$\{v^{-1} [a_o G Q + a_1 H Q^2]$$

$$+ [a_o Q P + a_1 Q^2 S + (a_o + a_1 Q) G + 2 a_1 Q H]$$

$$+ v [a_o P + a_1 Q (P + 2S) + a_1 (G + H)]$$

$$+ v^2 [a_1 (P + S)] \} .$$

(Assume G, H, μ are small)

$$\doteq v^{-1} [a_o G Q + a_1 H Q^2 - \mu a_o Q P - \mu a_1 Q^2 S]$$

$$+ [a_o Q P + a_1 Q^2 S + (a_o + a_1 Q) G + 2 a_1 Q H$$

$$- \mu a_o P - \mu a_1 Q (P + 2 S)]$$

$$+ v [a_o P + a_1 Q (P + 2S) + a_1 (G + H) - a_1 \mu (P + S)]$$

$$+ v^2 [a_1 (P + S)] .$$

As in Model I, we would like $k(v)/a^*v$ to be independent of v as far as possible Since the coefficient of v^{-1} is assumed small, this suggests, following Model I, we make the coefficient of v zero, i.e., set

$$a_1 = - 1 \text{ [arbitrary]}$$

$$a_o = - a_1 \frac{Q (P + 2S) + G + H - \mu (P + S)}{P} .$$

The corresponding solution to the same accuracy, in Model I would be

$$a_o = - a_1 \frac{Q(P + 2S)}{P} .$$

[The correction

$$a_o I_1 + a_1 I_2 = 1$$

must be made in both cases; see eq. (4.4)] . However, recall that P, S are defined in terms of a^* not a and a^* differs from a .

Thus the optimal strategy for P is modified.

13 Consequences of non-optimal policy in the one-stage speed game

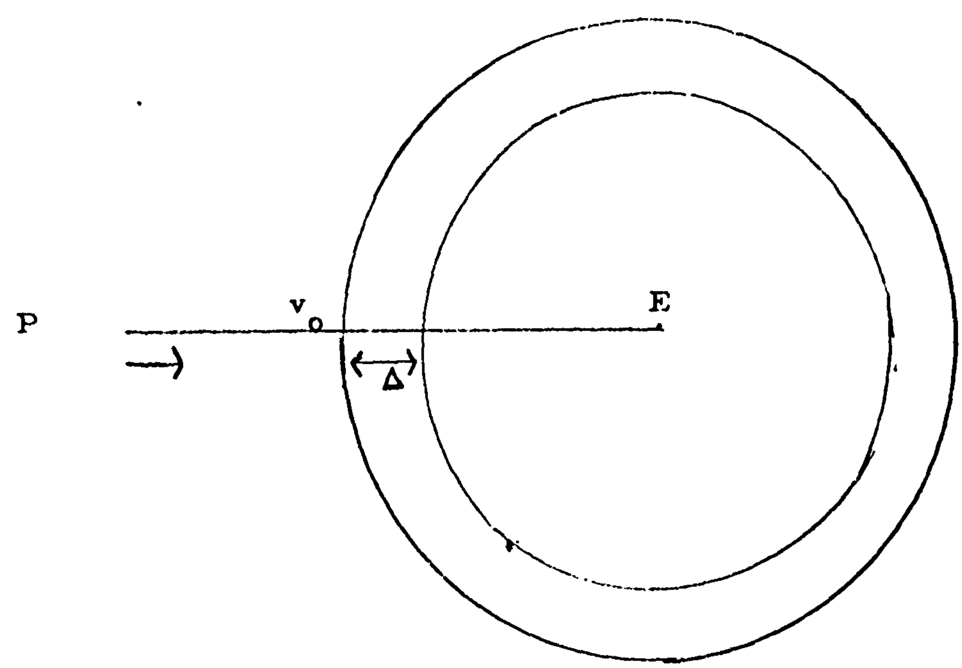
13.1 Introduction

We have developed optimal policies for a Destroyer vs. Submarine one-stage speed game. The question has been raised: What are the consequences should the Submarine (Evader, E) adopt a non-optimal policy? (In particular, if he adopts maximum speed v_o.) For realism, we may assume E chooses a speed v uniformly in the range $[v_o - \Delta, v_o]$ where $\Delta(= 0.1\, v_o$, say) is known to the Destroyer (Pursuer, P); we refer to this as the near-maximum speed policy.

13.2 Summary of Results

For the numerical example used, we show that choice of the non-optimal near-maximum speed policy by E leads to an increase in kill-probability in the ratio of 6:1 approximately.

13.3 Analysis



Suppose E adopts a speed in the range

(13.1) $[v_o - \Delta, v_o]$

and that Δ is known to P. Denote the annulus

$$v_o - \Delta \leq r \leq v_o \text{ by } A(\Delta) .$$

Then P adopts a "starting point" somewhere in the same range, which we will take as approximately $(v_o - (1/2)\Delta) = v$

The time of arrival of P at v is

(13.2) $$t_2 = \frac{D}{Q + v}$$

and the radius, in speed-space, of the "cookie cutter" detector is R/t_2. If

(13.3) $R/t_2 > \Delta$, i.e. $t_2 < R/\Delta = t_3$, say then only width Δ is contributed to the coverage of annulus $A(\Delta)$. This continues until $t = t_3$; then the "coverage" is R/t. Thus the total coverage of $A(\Delta)$ is approximately

(13.4) $$\int_{t=t_2}^{t_3} \frac{(q^2 - v^2)^{1/2}}{t} \Delta \, dt + \int_{t=t_3}^{\infty} \frac{(q^2 - v^2)^{1/2}}{t} \frac{R}{t} \, dt$$

$$= (q^2 - v^2)^{1/2} \left[\Delta \log (t_3/t_2) + R/t_3\right] =$$

$$
= \begin{cases} (q^2 - v^2)^{1/2} \; \Delta [\log (R / \Delta t_2) + 1] & \text{if } t_2 < R/\Delta \qquad (13.5) \\ (q^2 - v^2)^{1/2} \; (R/t_2) & \text{if } t_2 > R/\Delta \qquad (13.6) \end{cases}
$$

13.4 Numerical Example

As in Section 6, let $Q = 2.0$

$q = 2.0$

$R = 0.05$

$D = 2.0$ and take $\Delta = 0.1$

Then $v = 0.95,$

$t_2 = D/(Q + v) = 2/2.95 = 0.678$ ($> R/\Delta = 0.5$)

$R/t_2 = 0.0737$

$(q^2 - v^2)^{1/2} = (3.0975)^{1/2} = 1.760$

Since $t_2 > R\Delta$ we use formula (13.6), to obtain: total coverage of annulus $A(\Delta)$ is

$(1.760)\ (0.0737) = 0.1298$

The area of $A(\Delta)$ is $\pi\ (1^2 - 0.9^2) = \pi.(0.19)$

Thus $P(\text{Kill}).\ 2\pi = 2(0.1298)/0.19$

$= 1.366$

In Section 6, when E used optimal strategy, we obtained

$P\ (\text{Kill}).\ 2\pi = 0.221$

Thus an increase in the kill-probability in the ratio of 6:1 is observed.

Note that the ratio of areas of the speed circle utilized by E in these two strategies is

$$\frac{\pi (1^2 - 0.9^2)}{\pi (1^2)} = 0.19 = 1/5 \text{ approx.}$$

E concentrates his effort on 20% of the available "strategy-space" (i.e. speed circle).

Why is P able to improve his kill-probability by more than the 5:1 ratio? Because P is relatively a more efficient searcher near the rim of the speed circle. True, his transverse speed is 1.760 compared to 2.0 near the center of the circle, but his radius of detection in speed space, R/t, is 0.0737 in place of 0.50 at the center (where t = 1 hour approximately).

Unless there are reasons why the Evader should get as far away as possible, he risks a 6-fold increase in P(Kill) by adopting near-maximum speed, for the example considered.

REFERENCES

[1] John M. Danskin, "A Helicopter versus Submarine Search Game", Operations Research, v. 16 (1968), pp. 509-517.

CHAPTER III

DISCRETE TWO-PERSON GAME THEORY WITH MEDIAN PAYOFF CRITERION

John E. Walsh

DISCRETE TWO-PERSON GAME THEORY WITH MEDIAN PAYOFF CRITERION

ABSTRACT

The minimax concept for solution of discrete two-person games is based on expected value considerations and has a zero-sum condition for payoffs. This approach often is inapplicable due to strong violation of the zero-sum condition. By use of a different criterion, based on median value considerations, a two-person game theory can be developed that seems appropriate for situation which we call competitive. It is also applicable for a much broader class of practically important situations called median competitive. Consider cases where each player is either protective toward himself or vindictive toward the other player. A largest value P_I (P_{II}) occurs in the payoff matrix for protective player I (II) such that he can assure himself at least this payoff with probability at least 50 percent. A smallest value P_I' (P_{II}') occurs in the matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 50 percent, that player I (II) receives at most this payoff. For competitive and median competitive games, a player is simultaneously protective and vindictive. Values of P_I, P_{II}, P_I', P_{II}', and median-optimum strategies, are nearly always determined without great effort. This can be done by solution of zero-sum games (expected value basis) with identified payoff matrices containing only ones and zeroes. Deciding on payoff values is simplified for the median

approach. Except for P_I, P_{II}, P'_I, P'_{II}, it is sufficient to know the relative order of the values for each payoff matrix. The median a; ɔach has the strong practical advantage of being applicable even when payoffs in different matrices cannot meaningfully be added or subtracted (such as when only relative ordering is known for one or both matrices).

INTRODUCTION AND DISCUSSION

This paper presents a form of discrete two-person game theory that is based on median value considerations (motivated by the median estimation concept in statistics). Two extreme situations arise, depending on whether a player is acting protectively for himself or vindictively towards the other player. A protective player is interested in the largest payoff such that he can assure himself at least this value with a probability of at least 50 percent. A vindictive player is interested in the smallest payoff such that he can assure, with a probability of at least 50 percent, that the other player receives at most this value. A class of "Median Optimum" strategies can be defined, for either the protective or vindictive approach.

For a class of games called "competitive", there exist strategies which are simultaneously median optimum from the protective viewpoint and from the vindictive viewpoint. Consequently, for those games, identification of a median optimum strategy as protective or vindictive is unnecessary. In spite of certain parallels between the median-optimum viewpoint and the minimax viewpoint, a pure strategy may be median optimum without necessarily being a minimax strategy.

The first introductory material outlines the median approach and some of its properties. Then, some comparisons are made with the standard minimax form of two-person game theory.

Each player has a separate matrix that states the payoff he receives for each combination of a pure strategy of his with a pure strategy of the other player. Both of these matrices are considered to be known to each player. The pair of payoffs to the players that occurs for a given combination of a pure strategy for each player is an outcome of the game. The first value in an outcome is the payoff to player I and the second value is the payoff to player II.

Each player then imposes a preference-ordering on the outcomes. The preference-ordering of a protective player I is considered to be such that his own payoffs are nondecreasing and, by convention, the preference-ordering of the protective player II is such that his own payoffs are nonincreasing. In contrast, a vindictive player I imposes an ordering such that the payoffs to player II are nonincreasing, and a vindictive player II imposes an ordering where the payoffs to player I are nondecreasing. If there are tied values among the payoffs considered, there may be many alternative orderings satisfying any one of those conditions. A vindictive player could order within ties so as to be as advantageous to himself as possible. A protective player could order within ties so as to be as disadvantageous to the other player as possible. On the other hand, a protective player could order within ties so as to be as advantageous to the other player as possible, etc. Finally, more than one outcome could possibly have the same pair of payoffs. A player may select among the

possible orderings of such "double ties" on the basis of the strategy combinations corresponding to these outcomes. In all cases, each player (identified as protective or vindictive) chooses a sequence that is called his preferred sequence. The preferred sequences provide the basis for application of the median approach.

A game is said to be competitive if the outcomes can be sequence ordered so that the payoffs for player I are nondecreasing and the payoffs for player II are nonincreasing. The possible ordering of outcomes is unique when the payoffs of player I are strictly increasing or those of player II are strictly decreasing; then we assume that the preference orderings of the two players are the same. However, more than one eligible sequence order is possible when ties in payoff value occur for both players. That is, the same outcome value could possibly occur for more than one combination of strategies. Among these sequences (which are the same in values of outcomes), each player selects his preferred sequence on the basis of the strategy combinations corresponding to the pertinent tied outcomes and related probability considerations. If the preferred sequence is still not unique, it can be chosen arbitrarily.

A largest payoff P_I (P_{II}) occurs in the matrix for player I (II) such that he can assure at least this value with a probability of at least 50 percent. A smallest payoff P'_I (P'_{II}) occurs in the matrix for player I (II) such that player II (I) can assure, with a probability of at least 50 percent, that player I (II) receives at most this amount.

Let us consider determination of P_I (P_{II}) for a protective player I (II). There exists a subset of outcomes in the preferred sequence for protective player I (II) that consists of a determined outcome and all outcomes above (below) it. One of the outcomes in this identified subset can be assured with a probability of at least 50 percent, but such is not the case when the determined outcome is removed. A protective player is considered select his preferred sequence so as to minimize the number of outcomes in the identified subset. When more than one eligible subset has the minimum number of outcomes, a subset that has the highest assured probability is used for the preferred sequence. The value of P_I (P_{II}) is the payoff for player I (II) in the lowest (highest) outcome of his identified subset. Incidentally, one or more outcomes consecutively adjacent to the identified subset could also have payoff P_I (P_{II}) for player I (II).

Now, consider determination of the payoff P'_{II} (P'_I), in the other player's matrix, that is associated with a vindictive player I (II). There exists a subset of outcomes in the preferred sequence for vindictive player I (II) that is identified in the same way as was given for evaluation P_I and P_{II}. The value of P'_{II} (P'_I) is the payoff to player II (I) in the lowest (highest) outcome of the identified subset for player II (I).

Detailed statement of results for the general case occurs in the next section. However, some additional information about results for competitive games is given here. When each player has a pure median

optimum strategy, player I (II) can guarantee (100 percent probability) at least $P_I = P'_I$ ($P_{II} = P'_{II}$). When player I (II) has a pure median optimum strategy but player II (I) does not, player I (II) can guarantee at least P_I (P_{II}) and player II (I) can assure at least P'_{II} (P'_I) with a probability greater than 50 percent. When neither player has a pure median optimum strategy, player I (II) can assure that he receives at least P_I (P_{II}) with probability at least 50 percent, and that player II (I) receives at most P'_{II} (P'_I) with probability at least 50 percent.

It is instructive to consider some characteristics of the median of a probability distribution for a situation where both pla‧‧ ‧se mixed strategies. The median value of a distribution is not ne‧ ‧arily unique. That is, all the permissible values in an extensive interval can be medians of a probability distribution. This property can be conven‧‧nt. Suppose that the game is competitive and consider the set of median values for player I (II). Also, suppose that both players use mixed strategies that are optimum for the median approach. Then, all payoffs that are at least equal to P'_I (P'_{II}) and at most equal to P_I (P_{II}) are medians of the probability distribution of the payoff to player I (II). Thus, as would be expected in competitive situations, player I (II) seeks to maximize the upper payoff P_I (P_{II}) in his set of median values and to minimize the lower payoff P'_{II} (P'_I) in the set of median values for player II (I). The properties of a median allow this to be done simultaneously in such a way that

P_I (P_{II}) and P'_I (P'_{II}) can be far apart in an ordering of the payoff values for player I (II). It is thus possible for P_I to be in the upper payoff values for player I simultaneously with P_{II} being in the upper values for player II. In fact, this seems to occur for many kinds of combinations of payoff matrices for players I and II in competitive games.

Required information about payoff matrices is not very great when the median approach is used. It is sufficient to first determine the relative order (includes equality) of the values for each matrix, which determines the locations of P_I, P'_I, P_{II}, and P'_{II}. Deciding on values for P_I, P'_I, P_{II}, and P'_{II} completes the information that is required about the payoff matrices.

Perhaps the most attractive feature of the median approach is its ability to handle competitive games where payoffs from different matrices cannot be meaningfully added or subtracted. This permits use of a game theory approach for an important and extensive class of situations. In fact, many of the situations occurring in economics, and the social science areas (psychology, education, etc.) have matrices of this kind, maybe due to the fact that only relative order can be determined within a payoff matrix. However, situations of this class occur in virtually all areas where game theory is potentially useful (including military applications)

It is more difficult to determine the appropriat ness of the median approach when the game is not even roughly of competitive nature. Then, a low payoff for one player does not necessarily correspond to a high payoff for the other. Thus, the median payoff, say to player I, might be substantially different for the protective and vindictive situations. Also, cases where cooperation would increase the payoff to both players can occur. However, the median approach can be useful when the players are not allowed to communicate (so that a player only knows his own payoff matrix). Also, results like those developed for competitive games can be obtained for a rather broad class of situations that are termed median competitive.

The most general form of median competitive games, and corresponding properties, have not been determined yet and provide a subject for future investigation. An example of games that are median competitive, but not necessarily competitive, is given here. The example consists of all games that "generate" competitive games. A game is said to generate a competitive game if, for both players, there exist sequences that are eligible to be preferred sequences and for which the following two conditions are satisfied: First, the payoffs of player I (II) that are in outcomes above (below) the outcome determining P_I (P_{II}) are at least (most) equal to P_I (P_{II}), and the payoffs in outcomes below this outcome are at most (least) equal to P_I (P_{II}). Second, the payoffs of player I (II) that are

below (above) the outcome determining P'_I (P'_{II}) are at least (most) equal to P'_I (P'_{II}), and the payoffs in outcomes below this outcome are at most (least) equal to P'_I (P'_{II}). Then, new outcomes can be formed, by pairing the payoffs of player I with those of player II, that satisfy the requirements for a competitive game but leave the outcomes that determined P_I, P_{II}, P'_I, P'_{II} fixed and at the same sequence positions. This is done so that the groups of payoffs in the identified subsets for the competitive game are the same as the groups in these subsets for the original game. Since the results developed depend only on the outcomes that determine P_I, P_{II}, P'_I, P'_{II} and on the groups of payoffs in the identified subsets, the results for this competitive game also apply to the game from which it was generated.

Now let us compare the median approach with the minimax procedure where a zero-sum condition is imposed on the two payoff matrices and the criterion is the expected value of the payoff to player I. Discussion of expected value and median value properties occurs first.

The outcome that results when one or both players use a mixed strategy is a random value. This outcome can be identified by a representative property of its probability distributions. The mean of this distribution (expected value of the random outcome) is one representative property that could be considered. The distribution median (not necessarily unique) is another representiative property that is useful. Each of these properties has desirable and undesirable features. Neither has been shown to

be uniformly preferable to the other. Usually, choice of whether to consider the expected value or the median value is based on the utility and convenience aspects of the situation. If, for the situation considered, much more extensive results can be obtained for the median, statisticians seldom hesitate to consider it rather than the distribution mean.

Discrete two-person game theory is a case where median considerations seem to lead to more extensive results of a worthwhile nature than do expected value considerations. The median approach handles the rather broad class of competitive and median competitive games, including games where one or both payoff matrices have ordinal members. The minimax approach requires cardinal members in both matrices but still only applies to the small subclass of competitive games where the matrices at least roughly satisfy a zero-sum condition. Values must be determined for all (or nearly all) of the outcomes when the minimax approach is used. Except for a few payoffs (usually four, and never more than four) only relative order among the payoffs in each matrix must be determined for the median approach.

For games of a zero-sum type, it would seem that a combined use of the expected value criterion and the median approach could be desirable. That is, the strategy used by a player is at least approximately optimum in an expected value sense and also assures at least an identified payoff with a probability that has a lower bound not greatly below 50 percent.

The resulting median payment would ordinarily be less than P_I for player I and less than P_{II} for player II. Such strategies are especially desirable when values of payoffs are only roughly known but relative ordering is precisely known within each matrix. The determination of strategies with these combined properties is another subject for future investigation.

Only discrete games are considered here. However, extension of the median approach to continuous cases, and combinations of continuous and discrete cases, seems definitely possible and worthwhile. This extension is a further subject for future investigation.

The next section contains statements of how to determine P_I, P'_I, P_{II}, P'_{II} , and optimum strategies for each player. Also, properties of results using the median approach are stated more precisely. The final section contains the basis for the results (in terms of three theorems).

RESULTS

Let the payoff matrix for player I (II) be stated so that rows represent pure strategies for player I (II) and columns are pure strategies for player II (I). For all applications, a marking of some of the values in the payoff matrices is made initially, with this being done separately for each matrix. The case of a protective player is considered first.

For protective player I (II), first mark the position, in his matrix, of his payoff in the last (first) outcome of his preferred sequence of outcomes. Then do this for the next to the last (first) outcome for player I (II), etc. Continue consecutively in his preferred sequence of outcomes until the first time that this player can assure obtaining a marked value with probability at least 1/2. The value of P_I (P_{II}) is the last payoff marked in the matrix for player I (II). For competitive games, P'_I (P'_{II}) is the payoff to player I (II) in the last outcome that was marked in the matrix of player II (I).

Determination of P_I (P_{II}), and the corresponding pairs, is perhaps best accomplished by initially marking the matrix for player I (II) until the first time that two or fewer rows contain marks in all the columns. (The value of P_I (P_{II}) is greater than or equal to the last payoff marked in this manner, and can be greater; based on Theorem 1.) Next, working forward (backward) in the preferred sequence for protective player I (II),

remove the mark from the payoff (unique) that was marked last. Then, replace the remaining marked values with ones and replace all other payoffs in the matrix by zeroes. Consider this matrix of ones and zeroes to be for a zero-sum game with an expected value basis. Solve for the value of the game. If this game-value is less than 1/2, the marking is completed by again marking the payoff whose mark was removed (which also determines the corresponding pair). If the game-value is at least 1/2, continue in the same way (removing the mark from the last payoff that was considered among those still marked, forming a matrix with ones and zeroes, etc.). If the resulting game-value is less than 1/2 the payoff whose mark was last removed is marked again and the marking is completed. This marking procedure is continued until a game-value less than 1/2 occurs. (This procedure, and that in the next paragraph, are based on Theorem 2.) From the examples, it seems that P_I and P_{II} are often the payoffs that resulted in the first time that two or fewer rows contain marked values in all columns of the respective matrices.

The zero-sum game (matrix of ones and zeroes) that occurs for the final marking in evaluating P_I or P_{II} is also used to determine (protective) median optimum strategies for the player with that matrix. That is, an optimum strategy of this player for that game is also a median optimum strategy. In particular, consider the situation

for player I (II) when P_I (P_{II}) happens to be the payoff whose marking resulted in a pair of rows that contain marked values in all columns (but no fully marked row occurs). Examination of the zero-sum game shows that a mixed median optimum strategy for player I (II) consists in choosing one of the rows of this pair with probability 1/2 for each row.

For player I (II) vindictive, first mark the position in the matrix for player II (I) that is in the last (first) outcome of the preferred sequence of player I (II). Then do this for the next to last (first) outcome for player I (II). Continue consecutively in the preferred sequence for player I (II) until the first time that he can assure obtaining a marked value in the matrix of player II (I) with probability at least 1/2. The value of P'_{II} (P'_I) is the last payoff marked in the matrix for player II (I).

Determination of P'_{II} (P'_I) can be accomplished by initially marking the matrix for player II (I), according to the vindictive preferred sequence for player I (II), until the first time that two columns contain marks in all the rows. Next, remove the mark from the payoff that was marked last. Replace the remaining marked values with ones and all other payoffs by zeroes. Consider the resulting matrix to be for a zero-sum game and solve for the game-value. If this game-value is greater than 1/2, the marking is completed by again marking the payoff whose mark was removed. If the game-value is at most 1/2, continue in the same way with removal of another mark. If the resulting game-value is greater than 1/2, again mark the payoff whose mark was last removed

and the marking is completed. This marking procedure is continued until a game-value greater than 1/2 occurs. As for the protective case, it seems that P'_I and P'_{II} are often the payoffs that resulted the first time that two or fewer columns contained marked values in all rows.

The zero-sum game that occurs for the final marking of the matrix for player II (I) can be used to determine (vindictive) median optimum strategies for player I (II). That is, an optimum strategy of player I (II) for this game, that is based on the matrix for player II (I), is also a median optimum strategy. When P'_{II} (P'_I) happens to be the payoff that resulted in a pair of columns with marks in all rows (but no fully marked column occurs), a mixed median optimum strategy for player I (II) consists in selecting one of these two columns with probability 1/2 for each column.

Statement of results occurs next. Cases where pure median optimum strategies occur are considered first. In all cases, a protective player I (II) can guarantee himself at least P_I (P_{II}) by using the fully marked row in his matrix. A vindictive player I (II) can always guarantee that player II (I) recieves at most P'_{II} (P'_I), by using the fully marked column in the matrix for player II (I).

Now, consider the case where each player has a pure median optimum strategy. When a protective player I (II) uses the fully marked row in his matrix and a vindictive player II (I) uses the fully marked column

in the matrix for player I (II), player I (II) receives exactly $P_I = P'_I$ ($P_{II} = P'_{II}$) and player II (I) receives the payoff in his matrix that corresponds to the strategy combination for this row and column.

When protective players I and II both use fully marked rows, player I sometimes receives more than P_I and/or player II sometimes receives more than P_{II}. When vindictive players I and II both use fully marked columns (in the other player's matrix), player I sometimes receives less than P'_I and/or player II sometimes receives less than P'_{II}. When the game is competitive and each player has a pure median optimum strategy, $P_I = P'_I$ and $P_{II} = P'_{II}$ also, when each player uses his pure median optimum strategy, player I (II) receives P_I (P_{II}).

Now, consider the case where a pure median optimum strategy occurs for player I (II) but not for player II (I). First, suppose that player I (II) is protective. Then, player I (II) can guarantee himself at least P_I (P_{II}), and a protective player II (I) can assure himself at least P_{II} (P_I) with a probability of at least 1/2. Player I (II) can also guarantee himself at least P_I (P_{II}) against a vindictive player II (I), and player II (I) can assure that player I (II) receives at most $P_I = P'_I$ ($P_{II} = P'_{II}$) with a probability greater than 1/2 (Theorem 3). Next, suppose that player I (II) is vindictive. Then, player I (II) can guarantee that a protective player II (I) receives at most $P'_{II} = P_{II}$ ($P'_I = P_I$), and player II (I) can assure himself at least P_{II} (P_I) with a probability greater than 1/2 (Theorem 3).

Player I (II) can also guarantee that vindictive player II (I) receives at most P'_{II} (P'_I), and player II (I) can assure, with probability at least 1/2, that player I (II) receives at most P'_I (P'_{II}). Now consider competitive games. Then, $P_I = P'_I$ and $P_{II} = P'_{II}$ (Theorem 3). Player I (II) can guarantee that he receives at least P_I (P_{II}) and that player II (I) receives at most P_{II} (P_I). Player II (I) can assure that he receives at least P_{II} (P_I); also that player I (II) receives at most P_I (P_{II}) with a probability greater than 1/2.

Finally, consider the case where no pure median optimum strategy occurs for either player. Suppose that both players are protective. Then, player I (II) can assure at least P_I (P_{II}) with a probability of at least 1/2. When player I (II) is protective and player II (I) is vindictive, player I (II) can assure that he receives at least P_I (P_{II}) with probability at least 1/2 and player II (I) can assure that player I (II) receives at most P'_I (P'_{II}) with probability at least 1/2; these probabilities are exactly 1/2 when both players use mixed median optimum strategies. Next, suppose that both players are vindictive. Then, player I (II) can assure that player II (I) receives at most P'_{II} (P'_I) with a probability of at least 1/2. Now consider competitive games. Player I (II) can simultaneously assure, with probability at least 1/2, that he receives at least P_I (P_{II}) and that player II (I) receives at most P'_{II} (P'_I). When both players use mixed median optimum strategies, player I (II) receives at least P_I (P_{II}) with probability exactly 1/2 and at most P'_I (P'_{II}) with probability exactly 1/2.

Sometimes, a bound (for payoff values) that can be guaranteed does not differ much from the bound (determined using the median approach) that is only assured with a probability of at least 1/2. Then, the guaranteed bound would often be preferred. Existence of such situations should be considered in applications of the median approach (for payoff matrices with cardinal numbers).

BASIS FOR RESULTS

The procedure for determining P'_I, P'_{II}, and vindictive median optimum strategies can, with suitable interpretation be obtained directly from that for determining P_I, P_{II}, and protective median optimum strategies. Hence, verification for the protective case is sufficient.

The results for both players protective, or both vindictive, can be directly verified from the properties of the procedures for determining P_I, P_{II}, P'_I, and P'_{II}. This is also the case for one player protective and the other vindictive when both players have pure median optimum strategies or neither player has a pure median optimum strategy.

A competitive game can be considered to be a combination of the situation where player I is protective and player II vindictive with the situation where player I is vindictive and player II protective. Thus, to verify properties of competitive games, it is sufficient to present proof for the pertinent case(s) of one player protective and the other vindictive. Finally, it is to be noted that competitive players can have different preferred sequences in the sense of different combinations of strategies being associated with outcomes that have the same value. However, this causes no difficulties in derivations since the preferred sequences are the same with respect to the values of the outcomes.

The following three theorems contain the verification that is not evident from the properties of the procedures for determining P_I, P_{II}, P'_I, and P'_{II}.

Theorem 1. The procedure of marking payoffs (in his matrix) for a player until the first time that two or fewer rows contain marked values in all columns guarantees that occurrence of a marked value can be assured with a probability of at least 1/2.

Proof: First note that continued marking ultimately results in this situation. When one row becomes fully marked, the probability is unity that some one of the marked values can be assured by the player.

Next, suppose that a pair of rows is needed. When two mixed strategies $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are used (pure strategies are special cases, and there are r rows and s columns), the probability of the marked subset is

$$\sum_{i=1}^{r} p_i Q_i ,$$

where Q_i is the sum of the q's for columns that have marked payoffs in the i-th row. The largest value of this probability that the player can assure (by choice of $p_1, \ldots, p_r$) is

$$G = \min_{q_1, \ldots, q_s} (\max_i Q_i) .$$

Let i(1) and i(2) denote two rows that together contain marked payoffs in all columns. For any minimizing set of q's, both $Q_{i(1)}$ and $Q_{i(2)}$ are at most G, so that

$$2G \geq Q_{i(1)} + Q_{i(2)} \geq 1 ,$$

and a probability of at least 1/2 can be assured. This probability can exceed 1/2 but is exactly 1/2 when the unmarked payoffs are such that two columns contain unmarked payoffs in all rows (since analogously, the set of unmarked payoffs can be assured with a probability of at least 1/2). It is also exactly 1/2 when there are two columns that have an unmarked payoff in one of the two rows and are such that no row of the matrix has payoffs marked in both columns.

Theorem 2. A lower bound on the probability that a player can assure one of a specified subset of outcomes, and corresponding optimum strategies, can be determined by solution of a zero-sum game with an expected value basis. The payoff matrix for this zero-sum game has ones at the positions that correspond to the (pure) strategy combinations for the subset of outcomes, and zeroes elsewhere.

Proof: Let each player use an arbitrary mixed strategy (a pure strategy occurs as a special case). The expression for the expected payoff of the zero-sum game is also the expression for the probability that some one of the outcomes in the specified subset occurs.

Theorem 3. When protective player I (II) has a fully marked row in his matrix, but vindictive player II (I) does not have a fully marked column in this matrix, $P'_I = P_I$ ($P'_{II} = P_{II}$) ; also, player II (I) can assure that player I (II) receives at most P'_I (P'_{II}) with a probability greater than 1/2. Likewise, when vindictive player I (II) has a fully marked column in the matrix for protective player II (I), but player II (I) does not have a fully marked row in this matrix, $P'_{II} = P_{II}$ ($P'_I = P_I$) ; also, player I (II) can assure himself at least P'_{II} with a probability greater than 1/2.

Proof: Consider the outcome that corresponds to the last payoff marked for protective player I (II) and the outcomes that do not correspond to marked payoffs for player I (II). This set of outcomes can be assured with probability greater than 1/2 by vindictive player II (I). Otherwise, player I (II) would have terminated his marking procedure earlier. The payoffs for player I (II) in this set of outcomes are at most equal to P'_I (P'_{II}), with equality holding for the outcome corresponding to the last payoff marked for player I (II). This follows from the development of preferred sequences for competitive games. Also, since player I (II) has a fully marked row in his matrix, player II (I) cannot assure that player I (II) receives any payoff less than P'_I with nonzero probability. Thus, $P_I = P'_I$ ($P_{II} = P'_{II}$) and player II (I) can assure, with a probability greater than 1/2, that player I (II) receives at most P'_I (P'_{II}).

A similar verification can be given for the case of vindictive player I (II) having a fully marked column and protective player II (I) not having a fully marked row.

APPENDIX:*

EXAMPLES OF THE MEDIAN PAYOFF CRITERION AS APPLIED TO ZERO-SUM MATRIX GAMES

1. The preceding paper, "Discrete Two-Person Game Theory with Median Payoff Criterion," by John E. Walsh, introduced the concept of a median payoff criterion for a two-player matrix game. The present paper is intended to clarify that concept by providing some examples, which will be modifications of some well known matrix games, including "Matching Pennies" and "Paper, Stone, and Scissors."

2. Consider the game, "Matching Pennies," whose matrix is as shown. (We will use $A = (a_{ij})$ as the original payoff matrix.)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As usual, the payoff matrix shows payments from Player II to Player I. Thus, Player I is the maximizing player, and controls the row-index i of the matrix; Player II is the minimizing player and he controls the column-index j of the matrix.

In general, neither Player I nor Player II will have a strategy which is under all circumstances better than ("dominates") his other strategy. Thus, in general, each of the players must play a mixed strategy, which will prevent his opponent from predicting his behavior with certainty. In fact, the ordinary game-theoretic minimax solution to the game whose matrix is shown above, is that each player should

* -- by John P. Mayberry

play each of his strategies with probability 1/2. The result is an expected payoff of 1/2; i.e., denoting the minimax value of the game by v(A), we see that

$$v(A) = 1/2$$

Note that the expected value of 1/2 is obtained by averaging outcomes which must be either = 0 or = 1; consequently, the probability of an outcome equal to 1 must be 1/2, if both players play optimally in the minimax sense.

With these strategies, in other words, the minimizing player has guaranteed that he will achieve an outcome as low as 0 with probability 1/2 -- while the maximizing player has simultaneously assured that he will obtain, with probability 1/2, an outcome as great as 1. This example illustrates a case where the median - optimum value of the game is not uniquely defined, but is an entire interval -- here the interval between 0 and 1.

3. Suppose now that we modify the payoffs in the game of "Matching Pennies" somewhat, so that Player I, who is trying to match, wins only 3/4 when the match is on "Heads," but continues to win 1 when the match is on "Tails." The matrix A then appears as follows:

$$A = \begin{pmatrix} 3/4 & 0 \\ 0 & 1 \end{pmatrix} .$$

The minimax value of this game is now 3/7, which is smaller than 1/2, as it should be since we have decreased one of the outcomes to the maximizing player.

However, the median-optimum solution strategies remain unchanged -- each player should still play his two alternatives with probability 1/2 each. The median-optimum value of this game is now the interval from 0 to 3/4, instead of the interval from 0 to 1. If the payoff in one of the non-match cells (a_{12} or a_{21}) had been increased, in addition to the decrease in one of the matching cells, the interval which represented the median-optimum value would have been further narrowed, but the median-optimum strategies would have remained unchanged.

If we consider a new example, in which the decreasing "match" payoff and the increasing "non-match" payoff are almost equal, we might get a matrix such as this:

$$A = \begin{pmatrix} 1/3 + \epsilon & 0 \\ 1/3 - \epsilon & 1 \end{pmatrix}.$$

So long as $\epsilon > 0$ there will be an interval of uncertainty for the median value, and the median-optimum strategies will be unchanged. If ϵ becomes ≤ 0, however, it will happen that one player (in this case, I) will have a strategy (in this case, his second) which dominates his other strategy, so that the game will be strictly determined. The strictly-determined game will, of course, have an optimal pure strategy in the usual minimax sense and the same will necessarily be true in the median-optimum sense.

(This last fact can easily be seen by noting that the presence of a saddle point, which is equivalent with the conditions that the game be strictly determined, identifies a value such that each player can guarantee that the payoff is at least as favorable to him as that value with probability equal to 1 and not merely with probability equal to 1/2;

consequently, each player will possess a median-optimum strategy which consists of the pure strategy identified by that saddle point, and the median-optimum value will be equal to the minimax value.)

In fact, there are only two possibilities for a 2-by-2 median-optimum game: either the game <u>is</u> strictly determined, in which case its median-optimum value is equal to its minimax value; or the game <u>is not</u> strictly determined, in which case its median value will be the whole of an interval between the second-smallest and the second-largest of the four entries in the matrix.

4. Let us consider now the game whose matrix is shown here:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the game of the "pea under the walnut." Player I is trying to guess under which walnut the pea was left, and Player II is attempting to put the pea under one of the three walnuts so that Player I does not guess correctly. Ignoring the possibility of sleight-of-hand, which makes this game so interesting to professionals, we find the minimax value to be 1/3, and the optimal strategies for each player are the same -- to play each strategy with probability 1/3. But since the only outcomes possible are 0 and 1, the fact that Player I's minimax-optimal strategy gives him an expected value of 1/3 means that he cannot play so as to achieve a payoff = 1 with probability greater than 1/3; consequently 0 is the largest number which Player I can assure that he will attain with probability 1/2. Consequently, the

median-optimum value of this game is $v_{mo} = 0$. The presence of the 1's in the matrix is useless to the maximizing player, since he has no strategy which would permit him to achieve those outcomes with probability 1/2.

5. It is important to notice that the first and last of the above examples have a significant distinguishing property, apart from their symmetry -- namely, that the possible outcomes are only 0 and 1. Consider now a general matrix A, with entries either 0 or 1; this matrix defines a game whose outcomes consist of only 0 and 1's. The minimax value v(A) of that game must be either less than 1/2, equal to 1/2, or greater than 1/2.

If $v(A) = 1/2$, then Player I must possess a mixed strategy which guarantees to him an expected value of at least 1/2, regardless of what Player II does. But the only way to achieve an expected value of at least 1/2, when the only outcomes possible are 0 and 1, is to achieve the outcome 1 with probability at least 1/2. The same argument shows that the minimizing player can achieve an outcome of 0 with probability at least 1/2. As a consequence, the whole interval between 0 and 1 represents the set of median-optimum values of the game.

If, instead, $v(A) < 1/2$, then Player II must have a strategy that guarantees an expected outcome of less than 1/2; this implies that an outcome = 0 be achieved by II with probability greater than 1/2, regardless of Player I's choice of strategy. Consequently, the median-optimum of the game must be given by $v_{mo} = 0$.

Finally, if $v(A) > 1/2$, the median-optimum of that game must be given by $v_{mo} = 1$.

It is obvious that those three cases exhaust all the possibilities for matrix games with entries either 0 or 1.

6. With regard to the question of numerically calculating the median-optimum payoff for a given matrix game whose outcomes are not restricted to the values 0 and 1, two approaches may be taken. The first is conceptually much clearer and reveals the essence of the situation; the second promises to be more useful for computational purposes. In this section we describe the conceptually preferable approach.

If a number s is a median-optimal value of a matrix game defined by a matrix A, then Player II can assure that (with probability at least 1/2) no outcome $> s$ will be obtained, while Player I can assure that (with probability at least 1/2) no outcome $< s$ will be obtained. Let us define a derived matrix game $A' = (a'_{ij})$, whose entries are given by

$$a'_{ij} = \begin{Bmatrix} 0 \text{ if } a_{ij} < s \\ 1 \text{ if } a_{ij} > s \end{Bmatrix}$$

(Let us suppose, for the moment, that there are no entries $a_{ij} = s$; the contrary case will be considered below.) Now, Player II can, in the derived game, so play as to insure that the probability of a 0 will be at least 1/2, while Player I can so play that the probability of a 1 is at least 1/2. Consequently, we must have $v(A') = 1/2$. But if that condition holds for the value s, which was by assumption not equal to any entry in the payoff matrix, it must hold for an entire interval, ranging at least from the matrix entry next lower than s

to the matrix entry next higher than s.

We summarize the characteristics of the case considered above as follows: The entries in the original payoff matrix may be divided into two sets, which we can call the "low set" and the "high set;" each element of the low set must be smaller than each element of the high set; when the elements of the low set are replaced by 0 and the elements of the high set are replaced by 1, the minimax value of the resulting matrix game is 1/2; in this case, any number between the greatest payoff in the low set and the smallest in the high set will be a median-optimum value for the original game; it is not excluded that the value may actually range over a wider interval, if some of the largest elements of the low set of outcomes, and/or some of the smallest elements of the high set of outcomes, actually have no effect on the minimax value of the derived game.

In short, we see that it is possible to determine the median payoff value of a matrix game by considering a graph, like Figure 1 on the facing page, which shows for each proposed median-optimum value s (plotted on the abscissa) the minimax value v(A') of the derived game which would result if we replace all payoff entries smaller than s by 0, and all payoff entries larger than s by 1. Of course, when s is very small, all the entries of the derived matrix A' will be = 1, and the minimax value of the derived game will also be = 1; when s is very large, all the entries in the derived matrix A' will be = 0, and the minimax value of that derived game would be = 0. No change can occur in the minimax value of the derived game except when the proposed s traverses one of the entries in the original payoff matrix, at which time v(A') <u>may</u> change. The graph must, therefore, consist

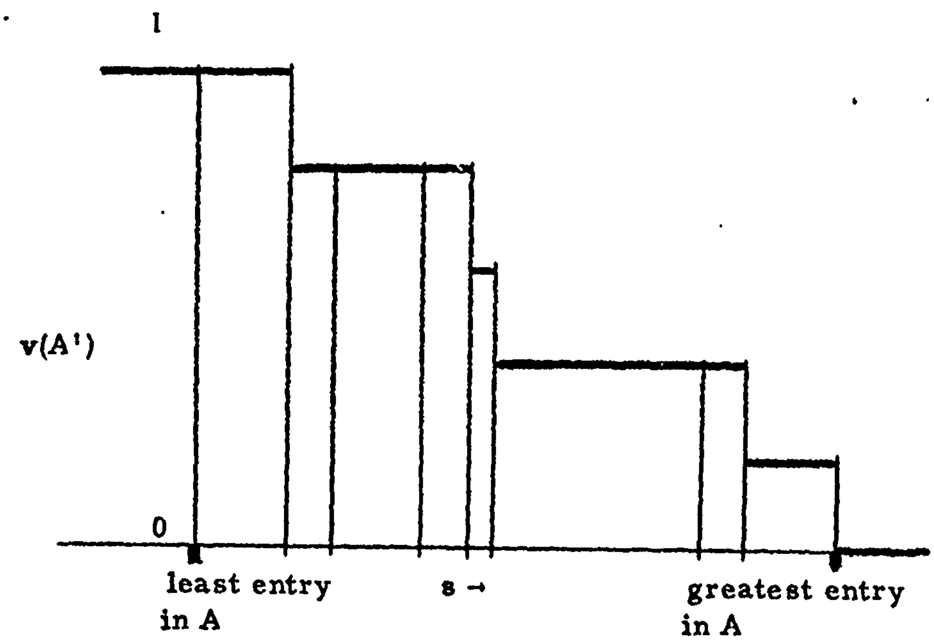


FIGURE I

Showing Minimax Value v(A') of the Derived Game A' for Various s

of a non-increasing function in a series of descending steps, each representing an interval in which v(A') does not change.

If one of those steps happens to be at the height 1/2, we have the situation described above, where partitioning the elements of the game matrix into a high set and a low set will provide a game whose minimax value is 1/2. It can very well occur, however, that none of those horizontal steps is at height 1/2. Then consider the last step greater than 1/2, and the next following step (which is less than 1/2). (Since the first step is preceded by an interval within which the minimax value is = 1, and the last step is followed by an interval within which the minimax value is = 0, there must be such a step.)

There will necessarily then be a unique s, separating the minimax values greater than 1/2 from those less than 1/2. That value s must be the median-optimum value of the original matrix and will be denoted by $v_{mo}(A)$.

7. Recall the concepts of minorant and majorant games, as introduced by von Neumann and Morgenstern in Ref [1]. The majorant game is the game in which Player II first chooses one of his pure strategies, and his decision is revealed to the maximizing player, who then makes his decision. This information gives a potential advantage to I, and the value of the majorant game, when only pure strategies are permitted to each player, represents an upper bound on the reasonable "values" for the game. This value is denoted by $\bar{v}$ (A), and is given by

$$\bar{v}(A) = \min_j \max_i a_{ij}$$

Because Player II can assure that the expected payoff will not exceed $\bar{v}$, Player II can never be forced to tolerate a higher value than v.

Analogously, we find the minorant game, whose value $\underline{v}$ is given by $\underline{v} = \max_i \min_j a_{ij}$; $\underline{v}$ represents a lower bound on what Player I could ever be forced to tolerate.

If $\underline{v} = \bar{v}$, the game is strictly determined, and possesses a saddle point; each player can choose an optimal pure strategy and the value of the game v(A) is the common value of $\underline{v}$ and $\bar{v}$.

$$v(A) = \bar{v}(A) = \underline{v}(A).$$

This argument is entirely ordinal -- that is to say, it depends only on ordinal relations among the elements in the payoff matrix, and not on any geometric or arithmetic relations among them. Consequently, the median payoff criterion could be applied to a matrix game which possesses a saddle point, and could not influence the decision of rationality; each player should still play for the saddle point.

Therefore, if the game possesses a saddle point, the median-optimal value will be equal to the usual minimax value, which will be equal to the value at that saddle point. The converse is not true; there exists a game which has no saddle point, but whose median-optimum value is equal to one of the entries in the matrix. An example is given by this matrix:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} ,$$

which is, in fact, simply the game of Stone, Paper, and Scissors, or Jan-Ken-Pon. The median-optimum value of this game is 0, because

$$-1 < s < 0 \text{ implies } v(A') = 1/3, \text{ while}$$
$$0 < s < 1 \text{ implies } v(A') = 2/3.$$

If, on the other hand, $\underline{v} < \bar{v}$, then an examination of the argument of section 6 above, shows that the median-optimum value must lie between them:

$$\underline{v} \leq v_{mo} \leq \bar{v}.$$

8. Finally, we present three other examples; the first and third were generated by entering random numbers in a 6-by-6 matrix. The first has numbers to three decimal places, and therefore, does not have any "ties" between entries in its payoff matrix. Under these circumstances, it is still an open question whether the median-optimum values will occupy an entire interval whenever the matrix does not have a saddle point. The second example was obtained by selecting numbers from 0 to 5 at random, and therefore contains many duplications among its outcome values. The median-optimum payoffs, the minimax payoffs, and the optimal strategies are shown in each case.

The first example is given by this matrix:

$$A = \begin{pmatrix} .130 & .611 & .895 & .731 & .398 & .856 \\ .065 & .084 & .894 & .738 & .971 & .636 \\ .781 & .004 & .222 & .431 & .064 & .980 \\ .855 & .053 & .472 & .265 & .985 & .044 \\ .904 & .051 & .216 & .597 & .202 & .261 \\ .685 & .060 & .863 & .667 & .336 & .364 \end{pmatrix}.$$

For this game, the minorant value $\underline{v}$ is given by

$$\underline{v} = \max_i \min_j a_{ij} = 0.130,$$

and the minorant value $\bar{v}$ is given by

$$\bar{v} = \min_j \max_i a_{ij} = 0.611.$$

The median-optimum strategies ξ_{mo} and η_{mo} are given by

$$\xi_{mo} = ([illegible], 0, 0, 1/2, 0, 0),$$

$$\eta_{mo} = ([illegible], 1/2, 0, 0, 0, 0),$$

and the median-optimum value consists of the interval from 0.130 to 0.611:

$$v_{mo} = (0.130, 0.611).$$

In the above example, v_{mo} is the entire interval between $\underline{v}$ and $\bar{v}$. That is not the case in general, and we present a modification of the above example to illustrate the point (a_{45} has been decreased from 0.985 to 0.385):

$$A = \begin{pmatrix} .130 & .611 & .895 & .731 & .398 & .856 \\ .065 & .084 & .894 & .739 & .971 & .636 \\ .781 & .004 & .222 & .431 & .064 & .980 \\ .855 & .053 & .472 & .265 & .385 & .044 \\ .904 & .051 & .216 & .597 & .202 & .261 \\ .685 & .060 & .863 & .667 & .336 & .364 \end{pmatrix}$$

For the above matrix A, it is still true that

$$\bar{v} = \min_j \max_i a_{ij} = 0.611, \text{ and that}$$

$$\underline{v} = \max_i \min_j a_{ij} = 0.130;$$

and the median-optimum strategies are also unchanged:

$$\xi_{mo} = (1/2, 0, 0, 1/2, 0, 0),$$

$$\eta_{mo} = (1/2, 1/2, 0, 0, 0, 0).$$

However, the median-optimum value is now a narrower range:

$$v_{mo} = (0.130, 0.398).$$

It is worth noting that a median-optimum matrix game (with determinate entries in the payoff matrix), even when zero-sum, has some of the characteristics of a non-zero-sum game. This happens because the median-optimum value may (and in general will) be different for the two players; in this last example, Player I can assure that the

payoff is at least 0.398 with probability at least 1/2, while Player II can assure that the payoff is at most 0.130 with probability at least 1/2. It will, in some cases, be possible for "collusion" to take place between the players; the ramifications here seem to be very complex.

Another co[illegible]sequence of t[illegible] [illegible]ema[illegible]ks is that a median-optimum game cannot be reduce[illegible] [illegible] rne[illegible] [illegible]s a minimax-optimum game can, Ref [2]. If we examine [illegible] [illegible]bmat[illegible]x of the above examples, which consists of those entries in this payoff [illegible]atrix which are chosen with non-zero probability in some pair of median-optimum strategies, we get the 2-by-2 submatrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} .130 & .611 \\ .855 & .053 \end{pmatrix}$$

<u>in both cases</u>; the median-optimum values of this 2-by-2 matrix game are (0.130, 0.611) in both cases. In the second example, in other words, the median-optimum values are not the same for that sub-matrix as for the whole matrix.

If there exists a concept in the theory of median-optimum games, which corresponds to the "kernel" of a minimax game, it seems that it will have to be a more complex notion than the original notion of kernel presented in Ref [2].

For our third example, we consider the matrix A given here;

$$A = \begin{pmatrix} 5 & 1 & 0 & 5 & 1 & 4 \\ 4 & 3 & 3 & 4 & 1 & 5 \\ 1 & 1 & 0 & 4 & 0 & 2 \\ 3 & 2 & 2 & 3 & 2 & 2 \\ 1 & 0 & 5 & 5 & 1 & 0 \\ 5 & 3 & 2 & 4 & 5 & 5 \end{pmatrix}$$

In this matrix,

$$\underline{v} = \max_i \min_j a_{ij} = 2\,,$$

$$\bar{v} = \min_j \max_i a_{ij} = 3\,,$$

$$\xi_{mo} = (0,\ 1/2,\ 0,\ 0,\ 1/2),$$

$$\eta_{mo} = (0,\ 0,\ 1/2,\ 0,\ 1/2,\ 0), \text{ and}$$

$$v_{mo} = (2,\ 3).$$

Note that this game can be reduced to a kernel; the matrix

$$\begin{pmatrix} a_{23} & a_{25} \\ a_{63} & a_{65} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$$

has median-optimum values between 2 and 3, just as the whole original matrix did.

The minimax strategies are given by

$$23\xi = (0,\ 11,\ 0,\ 0,\ 2,\ 10),$$

$$23\eta = (0,\ 8,\ 12,\ 0,\ 3,\ 0),$$

and the value by

$$v(A) = \frac{63}{23}\ .$$

9. A significant question on the subject of median-optimum games remains open. If the entry in row i, column j, of the outcome matrix is a random variable $\tilde{x}_{ij}$ having a continuous probability-distribution (with $E(\tilde{x}_{ij}) = a_{ij}$), rather than a determinate outcome, then the derived matrix A' will be given by

$$a'_{ij} = \text{prob}\,\{\text{the random variable } \tilde{x}_{ij} \text{ exceeds } s\}.$$

Since each a'_{ij} will then be a continuous, strictly-decreasing function of s, and since $v(A')$ is a continuous function of the a'_{ij}, it follows that

$v(A')$ is a continuous strictly-decreasing function of s.

Thus, there will be a unique value of s for which $v(A') = 1/2$ and

$v_{mo}(A)$ is unique.

Computation of v_{mo} in this case is another open question.

References:

[1] Morgenstern, O., Von Neumann, J., "Theory of Games and Economic Behavior," Ch. II, 2nd ed. (1947), Princeton University Press, Princeton

[2] Shapley, L.S., and Snow, R.N., "Basic Solutions of Discrete Games," page 27 of Contributions to the Theory of Games, edited by Kuhn, H.W., and Tucker, A.W. (Annals of Mathematics Studies, Number 24), Princeton, 1950.

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CHAPTER IV

MIXED STRATEGIES IN PRACTICE

John P. Mayberry

MIXED STRATEGIES IN PRACTICE

I. INTRODUCTION

The notion of mixed strategy has been of fundamental importance in the study of zero-sum two-player games ever since it was first introduced by Borel (ref. [1]), and has been the subject of much discussion, both in a formal mathematical context and in an informal practical context. Much of the mathematical argument has focused on the convenience and logical necessity of the concept, while much of the informal discussion has emphasized the impracticality of a mixed strategy in applications. (A common objection may be expressed along these lines: " practical man will never believe that he ought to make rational decisions by flipping a coin".) The two principal theses of this paper are, first, that the mathematical purposes for which the mixed strategy was designed can be fulfilled by a variety of practical means, apart from actual randomization; and, second, that a minimax solution (even if it is not a pure strategy) can be, and often is, put into effect by a practical player.

Although we assume that the reader has some familiarity with the basic notions of two-person zero-sum games, we shall briefly recapitulate the historical development of the mixed strategy to define our viewpoint, and shall then proceed to suggest how the <u>purposes</u> of the mixed strategy can be achieved without "coin-flipping".

These suggestions lead naturally to further suggestions on the use of specific techniques to achieve the benefits of minimax solutions, without the inconvenience of randomization. These conclusions and recommendations appear to be of considerable potential importance in a wide class of practical problems.

II. HISTORICAL SUMMARY

The principal purpose which is achieved by the device of the mixed strategy is to provide to a participant an explicit recommendation for action, which nevertheless does not permit that action to be predictable.

In the case of "Matching Pennies"*, the recommendation states that each competitor, ("player") ought to choose heads and tails equally often, and without pattern - in other words, ought to choose each alternative at random, with equal probabilities. It is certainly sufficient to achieve the minimax expectation if the selection of pure strategies is made at random. However, it is not necessary that the strategies be chosen at random, but merely that they be chosen in a way which cannot be predicted by the opponent. In fact, whenever one player attempts to implement a mixed strategy, he necessarily takes a "random number" (or more precisely a "pseudo-random number" as generated by some device) uniformly distributed between 0 and 1, and then chooses (for example) heads whenever that random number is less than one-half, tails otherwise. If that pseudo-random number is known to the opponent, then this mixed-strategy recommendation will also be transparent to the opponent, and the first player's actions will in fact be predictable by the opponent.

It appears that the critical consideration is not whether the choice is actually random**, but whether it can be observed or inferred by the

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*In the version of "Matching Pennies" discussed by Von Neumann and Morgenstern (ref. [2]), each player chooses either "Heads" or "Tails"; player I wins a penny from player II if the choices match, pays a penny to player II if they do not.

**One might pursue a complex side issue at this point, discussing the question of whether using a specified random sequence requires an additional randomization to determine which elements of the sequence (e.g., numbers below one-half as against numbers above one-half) should correspond to tails. One might also discuss the question of whether or not the opponent would know, even if he had access to the random sequence, just how it was being used - so that he could employ that knowledge to predict your behavior. These points, although of some interest, are not germane to our principal concern in this paper.

opponent. A deterministic sequence is quite satisfactory if we can be certain that the opponent does not know it, and a random sequence is totally unsatisfactory if the opponent can discover it.

III. THREE ASPECTS OF A SPECIFIC MIXED-STRATEGY SOLUTION

Whenever a mathematical or formal model of a real process, including a conflict situation, is formulated significant factors are ignored. Indeed, the modeling process consists exactly in deciding which of the infinite variety of real experience must be ignored, what small portion must be considered, and (often the easiest part of the modeling operation) how to incorporate the effect of those factors which must be considered. Therefore, a formal model, which is intended to produce recommendations, should not be judged a failure merely because the recommendations it produces are ambiguous and indefinite. On the contrary, such ambiguity - where the model recommends one of a certain set of actions - provides the opportunity for incorporation into the actual decisions of the many factors ignored in the model. From this viewpoint, a solution which turns out to be a mixed strategy represents three separate recommendations: first, that certain pure strategies should not be used; second, that the complementary set, of "good" pure strategies, may sometimes be used; and, third, that only certain probability-mixtures of the "good" pure strategies should be used. We shall discuss each of those recommendations in turn.

a. Avoidance of Certain Strategies

The implementation of the first portion of the recommendation, that certain strategies should not be used, is extremely easy in practice and very compelling in principle; there can be very little reason for ever choosing any one of those pure strategies which does not appear in any optimal (minimax) strategy. Any one of those "non-good" strategies would, if played, provide the opponent with the opportunity to gain more than "his fair share" - that is, more than he could obtain if both sides played the minimax strategies. In case a player has the strong conviction that his opponent is not actually playing a minimax strategy, he might then wish to take advantage of this fact by himself deviating from the set of good strategies. We will refer again to this possibility, but for the present need only remark than an opponent who appears to be deviating from the minimax strategy is either playing badly or else he is playing a different game.

(In particular, he may be playing a non-zero-sum game). In this paper, we are restricting ourselves to the zero-sum case. Note that the phrase "the opponent appears to deviate from the good strategy" suggests observation of the opponent's strategy over a number of repetitions of the game; it is clear that much game-theoretic argument is implicitly concerned with adoption of policies which could hold not only for an isolated play of the game but also for repeated sequences of identical games. In fact, a frequentist view of probability would require that the notion of mixed strategy be bound together with the notion of repeated games or repeated opportunities to play the same game. From a subjectivist or a synthetic view of probability, however, there need be no inconsistency in applying the notion of mathematical expectation, even though the game is assumed to be played only once.

b. Use of Certain Strategies

Now we turn to the second point, that determination of the set of all mixed-strategy solutions specifies which pure strategies may ever be chosen, with non-zero probability, in a minimax solution. It is one of the elementary consequences of the minimax theorem that any such pure strategy will provide the value of the game against any optimal strategy of the opponent. Therefore, if we are sure that our opponent will be playing his minimax strategy against us (or one of his minimax strategies - since uniqueness is not guaranteed), we may play any of the pure strategies which appear in any of our optimal mixed strategies. Within our model of the zero-sum two-person game, it is assumed that both players have exactly the same view as to what the game is; that assumption implies that there is no uncertainty in rules, outcomes, or payoffs except uncertainties which are exactly described by probabilities - in which case the probabilities must be known to both players. Consequently, it is only outside the model, and within the context of the real applications, that we can speak of things being known to one player and not the other. Of course, in real applications, it is not always clear that the two antagonists have exactly the same view of the game which is being played; judgment must be employed in assessing the recommendations made in this paper insofar as they

rest logically upon such conditions. To summarize this second point, if there is one of our pure strategies which appears with a non-zero probability in some optimal mixed strategy and which is for some reason more desirable to us than any other such pure strategy, and if we are willing to assume that the factor which makes it more desirable is absolutely unknown and unsuspected by our opponent, then we may with impunity play that pure strategy instead of the mixed strategy which the minimax theory would recommend.

c. Appropriate Probabilities

Addressing the third point, the relative probabilites with which we may play those pure strategies which do appear in some optimal mixed strategy for our side - we have already discussed the question of when we could reasonably choose (with probability = 1) one of those pure strategies. In a single play of the game, no meaningful inference can be drawn from the way in which an opponent has actually played, as to whether his strategy was pure or mixed. Along the lines of the previous discussion, one may possibly derive information by considering which policies would be rational for an opponent to adopt and which would not. If there is, for example, one strategy for our opponent which gives him the opportunity for a large gain and also the opportunity for a large loss, and which would have simply been described in advance of the discovery of the minimax solution as "a case for the psychologist" or a case where the knowledge of one's opponent or intuition were required, the minimax theorem provides a quite specific recommendation to either avoid that strategy or to play it or possibly to play it with a certain probability only. The basis for this recommendation is essentially metatheoretic in nature. That is to say, if one is seeking a rational policy and if the rational policy is expected to provide explicit recommendations of what one should do, then one must assume that the opponent also knows of the existence of this method for finding rational policies, and would be able to apply it to the rules of the game.

On the other hand, if we believe that our perception of some part of the rules of the game (either of the available alternatives, or of the payoffs, or of probabilities attached to chance moves) is different from the corresponding perception of our opponent, [illegible] we may attempt to "outguess" the opponent. For example, if we believe that he perceives the game as a zero-sum two-player game [illegible] with precisely defined rules known both to us and to him, [illegible] to assume that he is playing a strategy which would [illegible] in Γ. If we perceive the game as being a somewhat different game, Γ', [illegible] but nevertheless are willing to assume that [illegible] the game as Γ' has not occurred to our opponent [illegible] to choose that strategy which will provide the greatest payoff to us in Γ' under the assumption that our opponent will be choosing one of the mixed strategies which is optimal for him in Γ. If he has only one such optimal strategy, η, our choice is then clear; we choose the strategy which optimizes against η. If, however, η is not unique, it will generally happen that the payoffs resulting for us in Γ' will depend both on which optimal η was chosen by the opponent, and on which strategy we choose. If we then attempt to optimize within that set, it is hard to find a logical process which will permit us to take full advantage of our superior knowledge (we know the game is Γ', while he thinks it is Γ). We should, under those assumptions, probably treat the opponent's choice of an η from within that set of optimal strategies as a purposeless choice, and employ one of the conventional solutions to "games against Nature".

IV. "OUTGUESSING"

Because there are significant differences between the min-max in pure strategies and the max-min in pure strategies, and because the actual value of the game generally lies strictly between those limits, we would be in effect surrendering to the opponent an advantage if he did in fact perceive the possibility of our interest in Γ'. Although this discussion has mentioned what one player knows, what one player wants, what one player thinks the other player knows, etc., it should be noted that all these statements of knowledge can easily be made quite formal and rigorous. One especially important situation can occur when each opponent has a certain set of opinions about the desires of the other, and subjective probabilities as to which of those opinions is true. If the game is truly zero-sum, and if it is only to be played once, and if the players share a common subjective probability on the various alternatives, then the payoffs may be averaged, and the situation may be subsumed in the ordinary zero-sum case. That is to say, one can formally demonstrate that the minimax mixed strategies are the unique reasonable way in which a play should behave. This remains true if the game is repeated, provided that each player's state of information, about the other's true desires, does not change; consequently, the repeated zero-sum game with some uncertainties about the rules and payoffs does not in fact present a new problem.

There are, however, two directions in which this model can be expanded, in order to encompass a greater variety of realistic situations. First, the model may be extended to include non-zero-sum games; second, we may assume that players may learn from the earlier plays of the game some facts which may influence their behavior on later plays. Efforts to combine these two extensions have so far been unsuccessful.

In the non-zero-sum case, one may ask either "What are reasonable ways for the players to behave, if they are not allowed to negotiate?" or one may ask "What are reasonable ways for the players to negotiate?" We may note that the use of subjective probabilities, as described above, may lead to a non-zero-sum case if the two players do not use the same subjective probabilities.

It appears that competitive behavior, with no vestige of negotiation, must result in an equilibrium point if there is only one such. If there are several equilibrium points, the players may lose by being unable to cooperate on the same one. The "Prisoners' Dilemma" paradox shows that this situation, even when the results are unambiguous, is often quite unsatisfactory.

If we allow the players to negotiate, there are various assumptions possible about the permitted types of bargaining and negotiation. Several specific problems of this type have been handled by Harsanyi, Selten, and others, who have formalized this notion of "bargaining game", and proven theorems about existence and uniqueness, in case each side merely makes a single proposal (so that "cooperation" occurs if the two proposals are consistent), and also in case a sequence of offers and counter-offers may be made (so that the notion of "cooperation" must be interpreted much more broadly). This last model is extermely complicated, and the solution of significant special cases has not yet been accomplished.

If the repetitive zero-sum model is extended in the second direction, so as to encompass the situation where each player learns something from the behavior of his opponent in earlier stages of the game, then it may happen, as plays of the game are repeated, that both parties effectively accumulate total information as to the actual payoff; or alternatively it may happen that, even after a large number of plays, residual uncertainty remains. The "limiting" game appears similar to the basic case where the uncertainty, and the probabilities, are known to both players.

On the other hand, some heuristic notions can be formulated which provide useful guidance in case an opponent appears (in a sequence of plays of a game Γ) to be playing badly. Three explanations should be considered:

one, he is indeed stupid; two, he is intelligently playing some other game Γ'; or three, he is trying to "mousetrap" us by pretending one of the first two, waiting for us to deviate from the set of our optimal strategies, and then taking advantage of the deviation. Consideration of the second case is very difficult unless we can formulate explicit notions of reasonable possibilities for Γ', but a response to both the first and third can be combined under this policy:

> choose our strategy so as to optimize our expected payoff against his observed historical average strategy, subject to the restriction that we do not expose ourselves to a larger expected loss on any one play than we have gained through his non-optimal play in the past.

CHAPTER V

REJECTION OF DATA: OPTIMAL ESTIMATION IN COMPLICATED DISTRIBUTIONS

John P. Mayberry

1. INTRODUCTION: THE BASIC CASE

In this paper we shall consider several problems of estimation of parameters in statistical distributions. We intend these problems to be useful and reasonable generalizations of the more-usual case of independent samples from a normal (Gaussian) distribution.

Let us recall the solution to that common case. If we have a sample $X = \{x_i\}$ of N real numbers, $x_1, x_2, \ldots, x_N$, which are assumed to be independent observations from a Gaussian population of unknown mean μ and variance σ^2, then the MLE (maximum-likelihood estimates) of μ and σ^2 are

$$(1.1) \quad \begin{cases} \hat{\mu}_{MLE} = \bar{x} = \dfrac{\Sigma x_i}{N} , \\ \hat{\sigma}^2_{MLE} = \dfrac{\Sigma (x_i - \bar{x})^2}{N} = \dfrac{\Sigma x_i^2}{N} - \bar{x}^2 . \end{cases}$$

It turns out that the modification

$$(1.2) \quad \hat{\sigma}^2_{UB} = \frac{\Sigma (x_i - \bar{x})^2}{N-1}$$

has a smaller bias.

2. EXAMPLE: MIXTURE OF TWO NORMAL DISTRIBUTIONS

Our first substantive example concerns independent samples from a distribution which is believed to be a mixture of two normal distributions, with a common but unknown mean μ and different standard deviations σ_1^2 and σ_2^2 (both also assumed unknown). We may assume without loss of generality that $\sigma_1 < \sigma_2$, and denote by p the probability

that each one of the measurements comes from the first ($\sigma^2 = \sigma_1^{\,2}$) distribution. Then the likelihood-function for a single measurement x is:

$$\mathcal{L}_1 (x \mid \sigma_1, \sigma_2, \mu, p)$$

(2.1)

$$= \frac{p}{\sigma_1 \sqrt{2\pi}} \cdot \operatorname{nexp}\left(\frac{(x-\mu)^2}{2\,\sigma_1^{\,2}}\right) + \frac{(1-p)}{\sigma_2 \sqrt{2\pi}} \cdot \operatorname{nexp}\left(\frac{(x-\mu)^2}{2\,\sigma_2^{\,2}}\right),$$

where $\operatorname{nexp}(x)$ denotes $\exp(-x) = e^{-x}$.

Although there are probably no difficulties of principle in estimating these four parameters from a sample of moderate size, we can reasonably consider the parameters p and $k = \frac{\sigma_2}{\sigma_1} > 1$ as being known in advance, since that facilitates the derivation of approximate solutions to the MLE equations. Furthermore, the procedures we shall derive below are not very sensitive to the exact values assumed for p and k. With that assumption, the above expression (2.1) simplifies* to

$$\mathcal{L}_2 (x \mid \sigma, \mu) =$$

(2.2)

$$\frac{1}{\sigma \sqrt{2\pi}} \left(p \cdot \operatorname{nexp}\left(\frac{(x-\mu)^2}{2\,\sigma^2}\right) + \frac{(1-p)}{k} \cdot \operatorname{nexp}\left(\frac{(x-\mu)^2}{2\,k^2\,\sigma^2}\right)\right).$$

Now, if we have our set $\mathcal{X} = \{x_i\}$ of N independent samples from that distribution, we get the joint likelihood function in the form

* We have also written σ for σ_1.

$$\mathcal{L}_3(\mathcal{X} \mid \sigma, \mu) =$$

(2.3)

$$(2\pi)^{-\frac{N}{2}} \cdot \sigma^{-N} \cdot \prod_{i=1}^{N} \left(p \cdot \mathrm{nexp}\left(\frac{(x_i-\mu)^2}{2\sigma^2}\right) + \frac{1-p}{k} \cdot \mathrm{nexp}\left(\frac{(x_i-\mu)^2}{2k^2\sigma^2}\right)\right) .$$

Now we may develop the MLE by solving the equations

$$(2.4) \quad \begin{cases} \dfrac{\partial}{\partial\mu} \mathcal{L}_3(X \mid \sigma, \mu) = 0 , \\[2ex] \dfrac{\partial}{\partial\sigma} \mathcal{L}_3(X \mid \sigma, \mu) = 0 . \end{cases}$$

After some tedious algebra, we reach a conclusion which can be easily expressed with the aid of some new notation. Let

$$(2.5) \quad \begin{cases} \lambda = k^{-2} , \\[1ex] A_i = p \cdot \mathrm{nexp}\left(\dfrac{(x_i-\mu)^2}{2\sigma^2}\right) , \\[2ex] B_i = \left(\dfrac{1-p}{k}\right) \cdot \mathrm{nexp}\left(\dfrac{(x_i-\mu)^2}{2k^2\sigma^2}\right) , \\[2ex] W_i = \dfrac{A_i + B_i \cdot \lambda}{A_i + B_i} . \end{cases}$$

(Note that A_i, B_i, and W_i all depend on μ and on σ; we do not make that dependence explicit, in order to keep the notation simple.)

The maximum-likelihood estimates of μ and σ can now be expressed in terms of the W_i, recognizing that the W_i will in turn depend on the values assumed or computed for μ and σ. We have confidence that the procedure, if iterated a modest number of times by recalculation of the W_i and the μ and σ, will generally converge. We also develop below a useful crude approximation which can be used for an initial estimate of the W_i.

Using the notation of (2.5), the maximum-likelihood estimates can now be expressed as follows:

$$
(2.6) \quad \begin{cases} \hat{\mu}_{MLE} = \dfrac{\Sigma_i W_i x_i}{\Sigma_i W_i} \\[2ex] \hat{\sigma}^2_{MLE} = \dfrac{1}{N} \Sigma_i W_i (x_i - \mu)^2 \\[2ex] \qquad\quad = \dfrac{1}{N} \Sigma_i W_i (x_i^2 \quad \mu^2) \,. \end{cases}
$$

(As a consequence of the first equation for μ, the two expressions for σ^2 given in (2.6) can be shown to be equivalent.) Just as in the prototype case of the Gaussian distribution, bias in the estimate is reduced by employing an alternate estimator for σ_1^2, viz,

$$
(2.7) \qquad \hat{\sigma}^2_{UB} = \frac{\Sigma_i W_i (x_i - \hat{\mu})^2}{N - 1}
$$

rather than those of equations (2.6).

Now, we can interpret the W_i as a more precise answer to the frequent question of "when shall we reject an outlier, or an apparently wild observation?" (Of course, we have made assumptions which are more narrow than the usual ones, so it is very natural that our answers are more precise; but the relationship is surprisingly close.) Recall that λ, which is equal to k^{-2}, is a number smaller than 1 - generally much smaller, because the variance of $k^2 \sigma_1{}^2 = \sigma_2{}^2$ is intended to explain observations which would be very improbable in a Gaussian distribution with variance $\sigma_1{}^2$. Then, recalling that A_i and B_i are strictly positive, the last equation of (2.5) implies that W_i will be always strictly between λ and 1; when $A_i \ll B_i \lambda$, W_i is nearly equal to λ. We may approximate W_i by 1 whenever $A_i > B_i$, and by λ whenever $A_i < B_i \lambda$. Recalling the definitions given in (2.5) for A_i and B_i, we can deduce (after some algebra) a crude approximation to W_i, as follows:

$$\text{(2.8)} \qquad \text{whenever} \quad \begin{cases} \left|\frac{x-\mu}{\sigma}\right| < \sqrt{2\ln\frac{pk}{1-p}}\,, & W_i \approx 1\,; \\ \left|\frac{x-\mu}{\sigma}\right| > \sqrt{2\ln\frac{pk^3}{1-p}}\,, & W_i \approx \lambda\,. \end{cases}$$

We have prepared a small table of those expressions, which might be called the "accept" and "reject" limits respectively; since $\lambda \ll 1$, to include an observation with weight $W_i = \lambda$ is nearly the same as to exclude it altogether. Between those limits, which are expressed (like (2.8)) in terms of $\left|\frac{x-\mu}{\sigma}\right|$, there will be a "partial acceptance" - in other words, a value of W_i intermediate between 1 and λ.

The procedure which results from this set of hypotheses is not very different from the rules of thumb often used for rejecting "outliers"; the logical foundation for those results is of course much clearer.

TABLE I: Critical Values of $\left| \frac{x-\mu}{\sigma} \right|$ for Various Combinations of p and k .

p	k	"Accept"	"Reject"
0.9	2	2.5	2.9
0.9	10	3.0	4.3
0.9	100	3.6	5.1
0.9	1000	4.3	6.0
0.99	2	3.3	3.7
0.99	10	3.8	4.8
0.99	100	4.3	5.9
0.99	1000	4.8	7.1

3. EXAMPLE: MIXTURE OF SEVERAL NORMAL DISTRIBUTIONS

We extend the results of Section 2 above to the problem of estimating the smallest (and most probable) of several standard deviations. We have m distributions, all with a common mean μ , with standard deviations σ_j and probability of occurence p_j, for $j = 1, 2, \ldots, m$.

We define* $k_j = (\sigma_j / \sigma_1)$, for $j = 1, 2, \ldots, m$, and assume the k_j and the p_j known *a priori*. We shall use σ as a synonym for σ_1 , from now on.

Then (2.2) becomes

* Of course, $k_1 = 1$ always in this notation.

$$\mathcal{L}_2(x|\sigma,\mu) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{j=1}^{m} \left(\operatorname{nexp}\left(\frac{(x-\mu)^2}{2\,k_j^2\,\sigma^2}\right) \cdot \frac{p_j}{k_j} \right) , \tag{3.1}$$

and the analog of (2.3), when we have observed a set $X = \{x_i \mid i = 1,2,\ldots,N\}$ of N independent samples, is

$$\mathcal{L}_3(X|\sigma,\mu) = (2\pi)^{-\frac{N}{2}} \cdot \sigma^{-N} \cdot \prod_{i=1}^{N} \sum_{j=1}^{m} \left(\operatorname{nexp}\left(\frac{(x_i-\mu)^2}{2k_j^2\,\sigma^2}\right) \cdot \frac{p_j}{k_j} \right) . \tag{3.2}$$

Now again we wish to develop MLE, so we set

$$\left\{ \begin{aligned} &\frac{\partial}{\partial\sigma} \log \mathcal{L}_3(X|\sigma,\mu) = 0 , \\ &\frac{\partial}{\partial\mu} \log \mathcal{L}_3(X|\sigma,\mu) = 0 . \end{aligned} \right. \tag{3.3}$$

Explicitly, we can express the first of those two results as

$$0 = \frac{-N}{\sigma} + \sum_i \frac{\sum_j \left(\frac{p_j\,(x_i-\mu)^2}{k_j^3\,\sigma^3} \cdot \operatorname{nexp}\left(\frac{(x_i-\mu)^2}{k_j^2\,\sigma^2}\right)\right)}{\sum_j \left(\operatorname{nexp}\left(\frac{(x_i-\mu)^2}{2k_j^2\,\sigma^2}\right) \cdot \frac{p_j}{k_j} \right)} , \tag{3.4}$$

which is equivalent to

$$\sigma^2 = \frac{1}{N} \sum_i W_i \, (x_i - \mu)^2 \tag{3.5}$$

where

$$W_i = \frac{\sum_j A_{ij} \cdot k_j^{-2}}{\sum_j A_{ij}} \, , \tag{3.6}$$

and

$$A_{ij} = \frac{p_j}{k_j} \, \mathrm{nexp} \left(\frac{(x_i - \mu)^2}{2k_j^2 \, \sigma^2} \right) \, . \tag{3.7}$$

The second equation of (3.3) then becomes

$$0 = \sum_i (x_i - \mu) \cdot \frac{\sum_j \frac{p_j}{k_j^3} \, \mathrm{nexp} \left(\frac{(x_i - \mu)^2}{2k_j^2 \, \sigma^2} \right)}{\sum_j \frac{p_j}{k_j} \, \mathrm{nexp} \left(\frac{(x_i - \mu)^2}{2k_j^2 \, \sigma^2} \right)} \tag{3.8}$$

from which can be derived

$$\mu = \frac{\Sigma_i \, x_i \cdot W_i}{\Sigma_i \, W_i} \, . \tag{3.9}$$

Now we note that

$$(3.10) \qquad W_i = \frac{A_{i1} + \Sigma_{j=2}^{m} (A_{ij} \cdot k_j^{-2})}{A_{i1} + \Sigma_{j=2}^{m} A_{ij}} .$$

If we compare this with equations (2.5), we see that we could make the computations here as simple as those of (2.5) if we defined $A_i = A_{i1}$,

$$(3.11) \qquad A_i = A_{i1} ,$$

$$(3.12) \qquad B_i = \Sigma_{j=2}^{m} A_{ij} ,$$

$$(3.13) \qquad \lambda_i = \frac{\Sigma_{j=2}^{m} (A_{ij} \cdot k_j^{-2})}{B_i} .$$

(Th: only real complexity is that λ_i depends on i in this case.)

In fact, each λ_i is a weighted average of the k_j^{-2} . To get a crude procedure corresponding to the procedure (2.8), we may simply drop the exponential term of (3.7), and define

$$(3.14) \qquad \lambda = \frac{\Sigma_{j=2}^{m} (p_j \cdot k_j^{-3})}{\Sigma_{j=2}^{m} (p_j \cdot k_j^{-1})} ,$$

$$(3.15) \qquad k = \lambda^{-\frac{1}{2}} \doteq \left(\frac{\Sigma_{j=2}^{m} (p_j \cdot k_j^{-1})}{\Sigma_{j=2}^{m} (p_j \cdot k_j^{-3})} \right)^{\frac{1}{2}} ,$$

and

(3.16) $$p = p_1 ,$$

and use the critical values developed as in Table I, above, to determine initial weights and "rejection levels".

4. Bias Affecting Several Observations

More complex than the cases considered above is the case where a common effect may bias several measurements. We treat first the case where an m-by-n matrix of elementary measurements is taken. We assume that each of the m rows consists of a set of n independent samples from a normal distribution with variance σ^2 and a common mean; with probability p that row mean is μ, and with probability (1-p) it is $\mu' \neq \mu$. (This is intended to be an approximation to the result of an anomaly in sonar transmission paths - some of our sets (rows) of measurements are estimating the correct quantity, some are estimating something else.)

Then the probability of a result X_{ij} as the j'th measurement in the i'th row is

$$(4.1) \qquad \frac{1}{\sigma\sqrt{2\pi}} \text{ nexp}\left(\frac{(X_{ij} - \mu_i)^2}{2\sigma^2}\right) \quad ,$$

where μ_i, the mean in the i'th row, is either μ (which occurs with probability p) or μ' (which occurs with probability (1-p)).

The likelihood $\mathcal{L}'_{II}$ of a set

$$(4.2) \qquad \chi = \{X_{ij}\}$$

of sample values is then given by

$$
(4.3)\quad
\left\{
\begin{aligned}
&\mathcal{L}'_{II}\;(\mathcal{X}|\sigma,\mu,\mu',p) = \\
&\prod_{i=1}^{m}\left(p\prod_{j=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\;\text{nexp}\left(\frac{(X_{ij}-\mu)^2}{2\sigma^2}\right)\;+\right.\\
&\qquad\left.(1-p)\prod_{j=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\;\text{nexp}\left(\frac{(X_{ij}-\mu')^2}{2\sigma^2}\right)\right)\\
&= (2\pi)^{-\frac{mn}{2}}\;\sigma^{-mn}\prod_{i=j}^{m}\left(p\;\text{nexp}\left(\frac{\Sigma_j\;(X_{ij}-\mu)^2}{2\sigma^2}\right)\right.\\
&\qquad\qquad\left.+\;(1-p)\;\text{nexp}\left(\frac{\Sigma_j\;(X_{ij}-\mu')^2}{2\sigma^2}\right)\right).
\end{aligned}
\right.
$$

Neglecting the constant first factor, we will get the Maximum-Likelihood Estimates by maximizing, instead of $\mathcal{L}_{II}$, the quantity

$$
(4.4)\quad \mathcal{L}_{II} = \sigma^{-mn}\prod_{i=1}^{m}\left(p.\text{nexp}\left(\frac{\Sigma_j(X_{ij}-\mu)^2}{2\sigma^2}\right) + (1-p)\;.\;\text{nexp}\left(\frac{\Sigma_j(X_{ij}-\mu')^2}{2\sigma^2}\right)\right).
$$

Assuming p known in advance, we must set

(4.5)
$$\begin{cases} \frac{\partial}{\partial\sigma} \log \mathcal{L}_{II} = 0 , \\ \frac{\partial}{\partial\mu} \log \mathcal{L}_{II} = 0 , \text{ and} \\ \frac{\partial}{\partial\mu'} \log \mathcal{L}_{II} = 0 . \end{cases}$$

Define

(4.6)
$$\begin{cases} S_i = \Sigma_j (X_{ij}-\mu)^2 , \\ S_i' = \Sigma_j (X_{ij}-\mu')^2 , \\ Q_i = p \cdot \operatorname{nexp}\left(\frac{S_i}{2\sigma^2}\right) , \text{ and} \\ Q_i' = (1-p) \cdot \operatorname{nexp}\left(\frac{S_i'}{2\sigma^2}\right) . \end{cases}$$

We can then conclude from (4.5) that

$$
(4.7) \quad \begin{cases}
\dfrac{mn}{\sigma} = \displaystyle\sum_{i=1}^{m} \left(\dfrac{S_i \cdot \sigma^{-3} \cdot Q_i + S_i' \cdot \sigma^{-3} \cdot Q_i'}{Q_i + Q_i'} \right) \\[2ex]
0 = \displaystyle\sum_{i=1}^{m} \left(\dfrac{\dfrac{\Sigma_j (X_{ij} - \mu)}{\sigma^2} \, Q_i}{Q_i + Q_i'} \right), \\[2ex]
0 = \displaystyle\sum_{i=1}^{m} \left(\dfrac{\dfrac{\Sigma_j (X_{ij} - \mu')}{\sigma^2} \, Q_i}{Q_i + Q_i'} \right).
\end{cases}
$$

Define

$$
(4.8) \quad \begin{cases}
R_i = \dfrac{Q_i}{Q_i + Q_i'} \quad \text{, and} \\[2ex]
R_i' = 1 - R_i = \dfrac{Q_i'}{Q_i + Q_i'} .
\end{cases}
$$

Then equations (4.7), after some algebra, imply respectively

$$
(4.9)\quad \begin{cases}
\sigma = \sqrt{\dfrac{\Sigma_{i=1}^{m} \quad (S_i R_i + S_i' R_i')}{mn}} \, , \\[2ex]
\mu = \dfrac{\Sigma_i R_i X_{i.}}{\Sigma_i R_i} \, , \text{ and,} \\[2ex]
\mu' = \dfrac{\Sigma_i R_i' X_{i.}}{\Sigma_i R_i'} \, , \text{ where} \\[2ex]
X_{i.} = \dfrac{\Sigma_j X_{ij}}{n} \, .
\end{cases}
$$

This algebra has a natural intuitive appeal; R_i is the posterior probability that the measurements of the i'th set have come from the distribution with mean μ, and $(1-R_i)$ is the posterior probability that those of the i'th set have come from the other distribution.

It is important to note that these equations are necessary, but not sufficient; it is quite possible that alternative choices could give local maxima of the likelihood function.

CHAPTER VI

A SIMPLE I-GAME

Francis M. Sand

A SIMPLE I-GAME

As a case study of the application of I-game methodology to ASW detection problem we will set up a scenario involving two alternative missions for the Red submarines. When it is assumed that Blue does not have prior information as to which of the missions Red is assigned, the conflict can be modeled as an I-game by having Blue act as a Bayesian. This means that he must attach prior probabilities p and 1-p to the two missions (which are assumed to affect the game payoffs substantially) and then optimize his expected payoffs. It has been shown [1] that this is equivalent to a game in which a unique "chance" move first determines the payoffs according to the same mixture (p, 1-p); then the Bayesian players choose their strategies. One or both may be ignorant of the outcome of the chance move. We assume that only Blue is ignorant.

Our scenario follows. Red will attempt to transit a large rectangle of ocean with T hours under one of two alternative game conditions: (i) The mission is essentially reconnaissance and a Red submarine must try to obtain information and return to base undetected. If the Red submarine is detected, its information becomes worthless. (ii) The mission requires that one of several Red submarines must come within range of a Blue shore target, preferably undetected. As many as possible of the Red submarines should evade detection in order to maximize the benefits of the mission. We now describe two simple strategies for each side.

Red's physical strategies: R_1 = Send one or two submarines and take evasive actions for maximum security.

R_2 = Send several submarines, some of which are intended to "blast through" even at considerable risk of being detected.

Blue's physical strategies: B_1 = Deploy ASW forces in the area so as to obtain maximal pursuit capability when a possible target signal is received.

B_2 = Defensive deployment which is best against R_2.

Now there are two games which we will label G_1 and G_2. Suppose the payoffs (as measured in terms of a weighted sum of detection and counterdetection probabilities) are as follows:

G_1

	B_1	B_2
R_1	1	-1
R_2	1/2	1

G_2

	B_1	B_2
R_1	1	-1
R_2	-1	0

The values of these two games are: $v(G_1) = 3/5$ and $v(G_2) = -1/3$.

Optimal mixed strategies are: $(1/5R_1,\ 4/5R_2)$ and $(4/5B_1,\ 1/5B_2)$ for G_1

and $(1/3R_1,\ 2/3R_2)$ and $(1/3B_1,\ 2/3B_2)$ for G_2

Red knows which game he must play; Blue must guess. If Blue has a priori probability p for G_1 being the true state of the world, and of course $1-p$ for G_2, and if he cannot avoid letting Red know the value of p, the Bayesian I-game model suggests the following reduction: both might play a new zero-sum game $G^*(p) = pG_1 + (1-p)G_2$ which has payoffs as follows:

	B_1	B_2
R_1	1	-1
R_2	$3p/2-1$	p

The value of $G^*(p)$ is $v^* = \frac{5p-2}{6-p}$ and the optimal mixed strategies are

$(\frac{1-p/2}{3-p/2}R_1,\ \frac{2}{3-p/2}R_2)$ and $(\frac{1+p}{3-p/2}B_1,\ \frac{2-3p/2}{3-p/2}B_2)$. As can be seen by substitution, when $p = 0$ the game $G^*(p)$ becomes G_2, and when $p = 1$, it becomes G_1; in both cases the value and optimal strategies are the same as given above. However, in the absence of any prior information, Blue would have to play indifferently, which means: $p = 1/2$. The resulting value and optimal strategies are: $1/11$, $(3/11R_1,\ 8/11R_2)$ and $(6/11B_1,\ 5/11B_2)$. If we assume that Red also plays as a Bayesian with prior prob-

ability $p = 1/2$, and that he does not know whether G_1 or G_2 is actually in force,* then these are the correct quantities for an optimal solution of the I-game. However, when we allow Red to use his information and differentiate his strategies according as G_1 or G_2 is in force, we will show that he can do better. The precise interpretation of the policy is a matter for careful study by the operational command, but one notices that, however the mixed strategies may be interpreted, the range of the weights attached to Blue's first strategy, B_1, is quite large: 4/5, 6/11, 1/3 for $p = 1$, 1/2, 0 respectively. Inasmuch as B_1 and B_2 involve substantially different deployments of the ASW forces, the three policies will most likely result in noticeably different actions.

The fact that Red knows which of G_1 and G_2 is being played, and Blue does not, may be represented in a 4 x 2 matrix game, $\overline{G}(p)$. The pure strategy choices for Red are expanded to four by allowing him to act differently according as the game is G_1 or G_2. The same option is <u>not</u> available to Blue.

$\overline{G}(p)$

	B_1	B_2
R_{11}	1	-1
R_{12}	2p-1	-p
R_{21}	1-p/2	2p-1
R_{22}	3p/2-1	p

<u>Red's Strategy Choices</u>

$R_{ij} \equiv$ "Choose R_i if G_1, R_j if G_2 is the true game."

---

* Admittedly a strange assumption in our example; we treat this case only for the sake of completeness of the discussion of the game-theoretic method.

The solution of $\overline{G}(p)$ is as follows:

If $0 \le p \le 4/5$, $\breve{v}(p) = (-2 + 7p)/6$, and
Red's optimal strategy is $\left[(0)\, R_{11},\ (0)\, R_{12},\ \left(\frac{1-p/2}{3-3p}\right) R_{21},\ \left(\frac{2-5p/2}{3-3p}\right) R_{22}\right]$;
Blue's optimal strategy is $\left[(1/3)\, B_1,\ (2/3)\, B_2\right]$.

If $4/5 < p \le 1$, $\breve{v}(p) = 3/5$, and
Red's optimal strategy is $\left[\left(\frac{5p-4}{5p}\right) R_{11},\ (0)\, R_{12},\ \left(\frac{4}{5p}\right) R_{21},\ (0)\, R_{22}\right]$;
Blue's optimal strategy is $\left[4/5,\ 1/5\right]$.

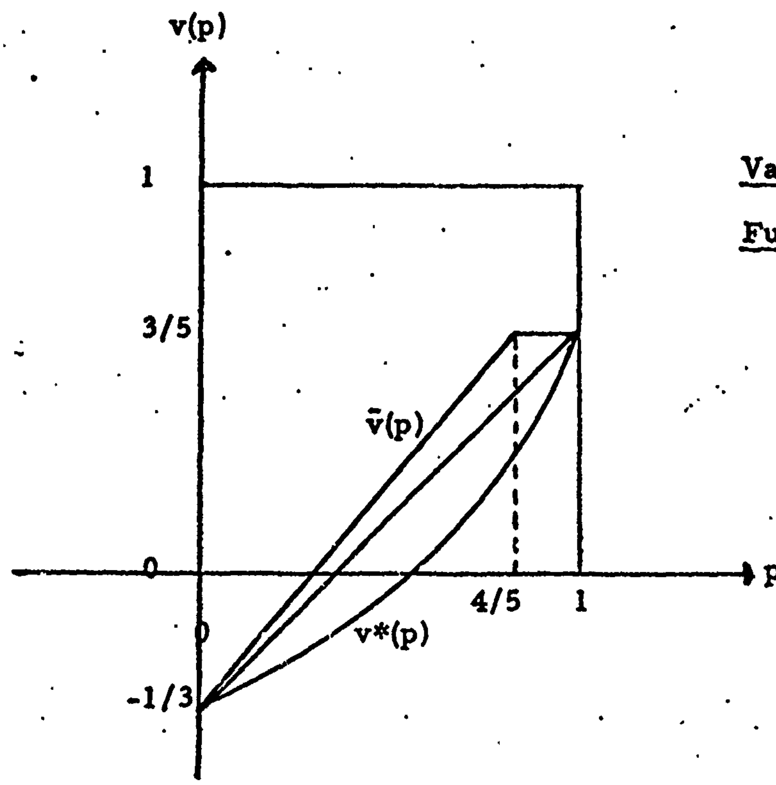

Figure 1

Value of I-game As a Function of Prior Probability

The upper graph in Figure 1 is the value of the game $\overline{G}(p)$; it is always higher than either $v^*(p)$ or the "reveal" line[1], which represents the value if Red were to announce to Blue which game was the true game after "chance" had chosen G_1 or G_2 with probabilities p and $1-p$. Thus we see that Red can take advantage of his superior information to gain a higher payoff by playing the I-game as if it were $\overline{G}(p)$. Only when $p = 0$ or 1 does he not make a positive gain compared with the other methods of optimizing. It is also worth noting that when $p < 4/5$, Red cannot achieve the "best" result of $v = 3/5$. A naive interpretation of the I-game might lead one to believe that Red merely has to play optimal strategies for whichever the true game happens to be to achieve this result. But this ignores the ability of Blue to infer what the true game is from Red's moves and act according to his inference. When p exceeds 4/5, however, Red can successfully play to the full advantage of his superior information.

A thorough analysis of the <u>repeated</u> I-game would show that Red can only improve on $G^*(p)$ to the extent of the "reveal" line mentioned above. His optimal strategy requires that he should (in the limit as the number of repetitions becomes indefinitely large) optimize for whichever of G_1 or G_2 is the true game. Blue would soon discover from Red's moves which is the true game, but Red would lose more if he attempted to conceal it from Blue. In this report we have treated the example as

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[1] The line joining (0, -1/3) and (1, 3/5).

a single instance of the I-game and thus have not emphasized the analysis of limit strategies for repeated games.

References

1. Harsanyi, John C., "Games with Incomplete Information Played by 'Bayesian' Players," Management Science, (Part I), Vol. 14 (1967), pp. 159-182 and (Parts II & III), Vol. 15 (1968), pp. 320-334 and 486-502.

APPENDIX*: An Alternative Viewpoint

To round out the paper, let us, in fact, look at $\overline{G} = (G_1, G_2)$ from Red's point of view.

We repeat the assumptions.

(i) Blue and Red know p (the a priori probability of nature choosing G_1)

(ii) Red knows "nature's choice" of which game G_1, or G_2, is being played, Blue does not.

We determine the value of the game, given Red knows which game is being played. Four cases arise:

(1) $p \leq 4/5$

Blue's strategy (identical for both games, of course) was noted previously: $[(1/3\ B_1, (2/3)\ B_2]$

Game :	G_1	G_2
Red's Strategy :	$[(0)R_1\ (1)R_2]$	$[(\frac{1 - p/2}{3 - 3p})\ R_1, (\frac{2 - 5p/2}{3 - 3p})\ R_2]$
Expected Pay-off:	5/6	-1/3

(2) $p > 4/5$

Blue's strategy (both games): $[(4/5)\ B_1, (1/5)\ B_2]$

Game :	G_1	G_2
Red's Strategy :	$[(\frac{5p - 4}{5p})\ R_1, (\frac{4p}{5})\ R_2]$	$[(1)\ R_1, (0)\ R_2]$
Expected Pay-off :	3/5	3/5

----

*by -- A. J. Truelove

Summarizing Expected Pay-offs:

G A M E	A priori P (Game)	Expected Pay-Offs $p \leq 4/5$	$p \geq 4/5$
G_1	p	5/6	3/5
G_2	1 - p	- 1/3	3/5

Given *a priori* probabilities for (G_1, G_2) of $(p, 1-p)$, Red's expected pay-off, before Red knows which game nature has chosen (and hence the value of the game (G_1, G_2) given p) is given by:

	Expected Pay-off from (G_1, G_2)
$p \leq 4/5$	$(5/6)\,p - (1/3)\,(1 - p) = (-2 + 7p)/6$
$p \geq 4/5$	3/5

Now consider the game G*, and suppose that Red is constrained to play the mixed strategy:

$$\left(\frac{1 - p/2}{3 - p/2}\ R_1,\ \frac{2}{3 - p/2}\ R_2\right),$$

even though he knows which of G_1, G_2 is being played. This situation would arise if Red did not wish to give Blue information on which

game is being played. (Of course, the pay-offs would have to be concealed from Blue also.)

Blue's strategy is $\left(\frac{1+p}{3-p/2} B_1, \frac{2-3p/2}{3-p/2} B_2\right)$ and the value of G^* (p) is $v^* = \frac{5p-2}{6-p}$.

As noted, Red can "reveal" to Blue which game is being played, by deviating from his best strategy for G^*.

We now calculate the value of G^*, given nature's actual choice (G_1 or G_2); i.e., we adopt Red's point of view.

Using the strategies for G^*:

Red Chooses	Blue Chooses	Probability of this pair of choices X $(3 - p/2)^2$	Pay-Off if game is: G_1	G_2
1	1	$1 - (1/2)p - (1/2)p^2$	1	1
1	2	$2 - (5/2)p + (3/4)p^2$	-1	-1
2	1	$2 + 2p$	1/2	-1
2	2	$4 - 3p$	1	0
	Expected Pay-Off		$-\frac{5p^2-4p-16}{(p-6)^2}$	$-\frac{5p^2-4p+12}{(p-6)^2}$

We verify the previous result that the overall expected pay-off (before nature has chosen G_1 or G_2, i.e., from Blue's point of view) is:

$$(p-6)^{-2} \left\{ (5p^2 - 4p - 16)p + (5p^2 - 4p + 12)(1-p) \right\}$$

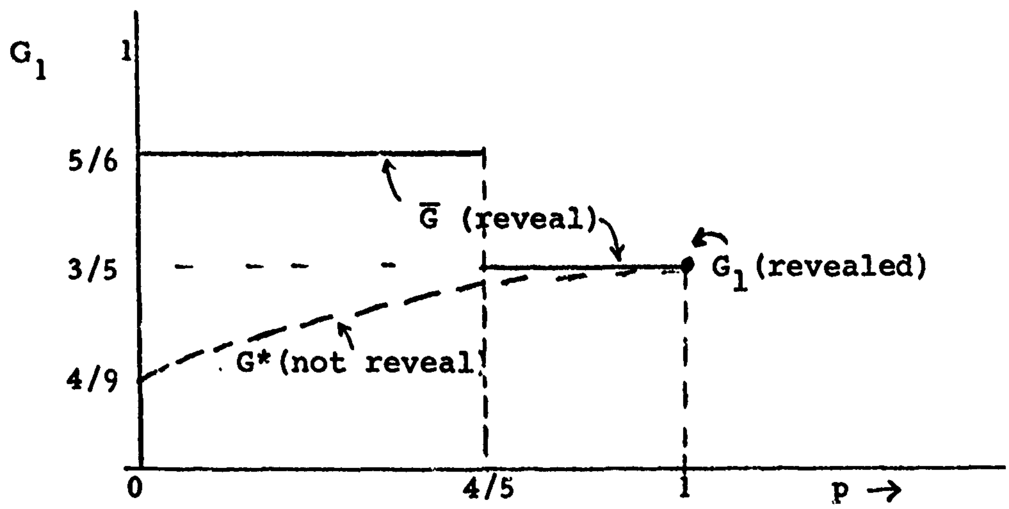
$$= \quad (5p-2)/(6-p) = v^*$$

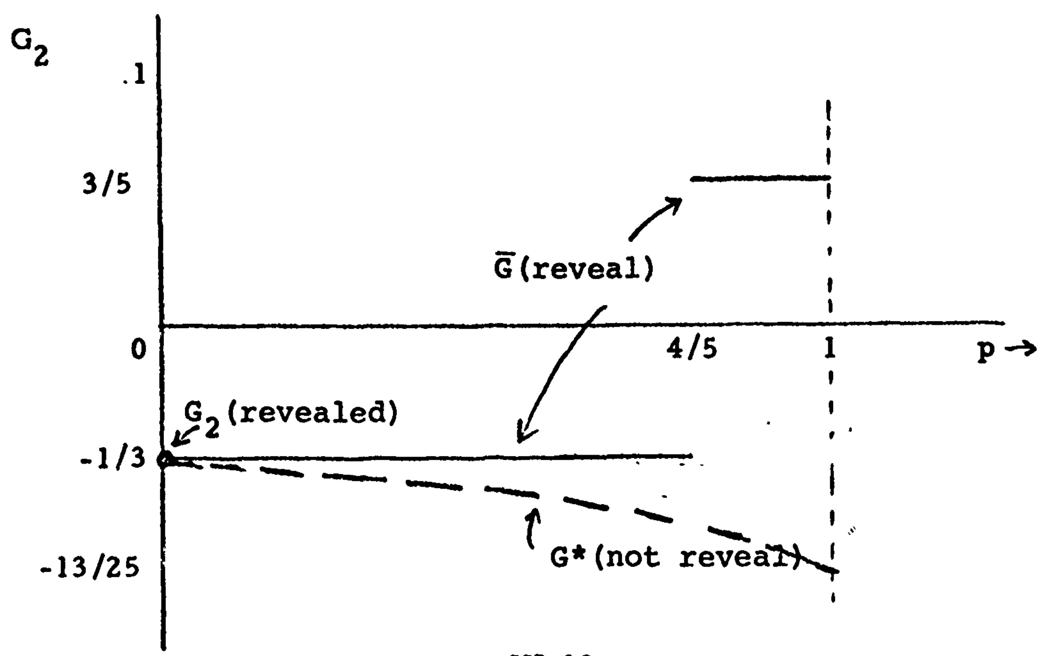
We now summarize:

(i) $\bar{G}$ - Red plays optimally, and (in the repeated game) eventually reveals which game is being played.

(ii) G* - Red's play is independent of which game is being played.

(iii) G_1/G_2 - both Red and Blue know the game (p takes only 1 value = 0 or 1).





Thus, even in the repeated game, Red always does best to reveal the game being played (even if the pay-offs were concealed from Blue). In conclusion, we note that this is not always true (see T. Saaty, "Mathematical Models of Arms Control and Disarmament," Wiley, 1968, p. 124, sec. 4.4; based on Aumann and Maschler, Game Theoretic Aspects of Gradual Disarmament in "Development of Utility Theory for Arms Control and Disarmament," Contract No. ACDA/ST-80 and 116 with MATHEMATICA, Princeton, New Jersey (1966).

CHAPTER VII

THE GENERALIZATION AND SOLUTION OF THE SMALL I-GAME DETECTION MODEL

Francis M. Sand

The Generalization and Solution of the Small I-game Detection Model

In the previous paper, an I-game formulation of ASW barriers and penetration strategies was presented and solved. The payoffs were intended to represent weighted combinations of the probability of detection of a submarine and the probability of counterdetection by a submarine. The differing utilities of the two events "detection" and "counterdetection", both within a mission, and as between two different missions, were not taken into account. This paper will extend the scope of applicability of the method by treating the payoffs as expected utilities. The assumption that both G_1 and G_2 are zero-sum games will be retained. A further assumption will be: that the probabilities of detection and counterdetection are the same in all similar strategic situations. In other words, these probabilities do not depend a priori on the mission type.

The framework of the I-game model, using the two assumptions of the foregoing paragraph, is represented below for the same two games G_1 and G_2 described in the previous case, allowing now for different utilities in the two situations however. Note that we do not allow the utilities to be affected by any considerations except (i) the mission type or game situation (G_1 or G_2) and (ii) the event typedetection or counterdetection. In the present working paper, no interaction between the event type and the mission type or between the time sequence of events and the mission is considered in the model.

Let u_i = U (counterdetection of a Blue ASW vehicle $\mid$ G_i)

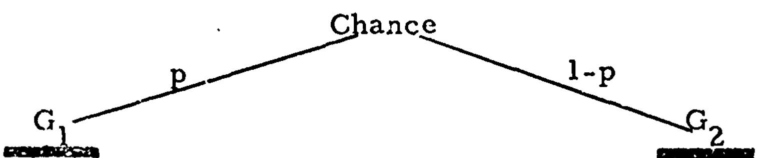
v_i - U (detection of a Red submarine $\mid$ G_i)

p_{ij} = Prob [counterdetection $\mid$ strategy choices R_i and B_j]

q_{ij} = Prob [detection $\mid$ strategy choices R_i and B_j]

with $i, j = 1, 2$.

In the game matrices which follows, the payoffs are expressed in utilities gained by Red who is the maximizer. Blue receives the negative of Red's gain.



	B_1	B_2
R_1	$u_1p_{11} + v_1q_{11}$	$u_1p_{12} + v_1q_{12}$
R_2	$u_1p_{21} + v_1q_{21}$	$u_1p_{22} + v_1q_{22}$

	B_1	B_2
R_1	$u_2p_{11} + v_2q_{11}$	$u_2p_{12} + v_2q_{12}$
R_2	$u_2p_{21} + v_2q_{21}$	$u_2p_{21} + v_2q_{21}$

For reasons which are intuitively obvious when the event types are related to the strategic choices, we will order the probabilities as follows:

$$p_{11} = q_{11} \qquad p_{12} > q_{12}$$

$$p_{21} < q_{21} \qquad p_{22} = q_{22}$$

$$p_{11} \geq p_{21} \qquad q_{11} \leq q_{21}$$

$$p_{12} \geq p_{22} \qquad q_{12} \leq q_{22}$$

To fix attention on a representative system of probabilities compatible with this ordering, consider the numerical values:

$$((p_{ij}, q_{ij})) = \begin{array}{c|c:c} & B_1 & B_2 \\ \hline R_1 & (1/2, 1/2) & (1, 0) \\ \hdashline R_2 & (1/4, 3/4) & (1/2, 1/2) \end{array}$$

With the unit weights* used in the previous paper we would have the games

$$\underline{G_1} \qquad \begin{array}{c|c:c} & B_1 & B_2 \\ \hline R_1 & 0 & 1 \\ \hdashline R_2 & -1/2 & 0 \end{array} \qquad\qquad \underline{G_2} \qquad \begin{array}{c|c:c} & B_1 & B_2 \\ \hline R_1 & 1/2 & 1 \\ \hdashline R_2 & 1/4 & 1/2 \end{array}$$

The utilities are generally unknown but may be ordered by analysis of the mission types. In fact, we will assume, for the present purposes, that $u_1 > 0$ and $v_1 < 0$ (for Red: the other way round for Blue). In case of mission type G_2, however, we will reason that the event type "counterdetection" is usually found only in association with "detection" but that "detection" may also occur alone at longer range. Accordingly, while both have negative utility for Red, in the light of his specific mission in G_2, we will assume $u_2 < v_2 < 0$. To fix

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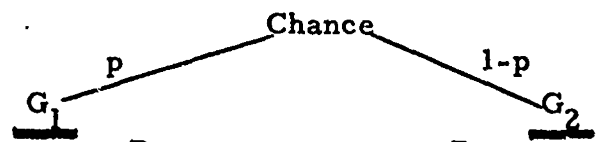
* $u_1 = u_2 = 1$, $v_1 = -1$, $v_2 = 0$ for example; unfortunately the actual G_1 and G_2 analyzed in the previous paper cannot be obtained from our general model due to a lack of adequate attention to realism in the earlier example.

on a definite analysis of the complete I-game model we will examine the case where $v_1/u_1 = -2$ and $v_2/u_2 = 3/4$. Thus we can write:

$$u_1 = u'_1 \quad ; \qquad v_1 = -2u'_1$$

$$u_2 = -4u'_2; \qquad v_2 = -3u'_2$$

with u'_1 and u'_2 being non-negative unknown constants. Now the I-game model must be represented as:



G_1	B_1	B_2
R_1	$(-1/2)u'_1$	u'_1
R_2	$(-5/4)u'_1$	$(-1/2)u'_1$

G_2	B_1	B_2
R_1	$(-7/2)u'_2$	$(-3)u'_2$
R_2	$(-13/4)u'_2$	$(-7/2)u'_2$

Solving the two games G_1 and G_2, we observe that G_1 has a saddle-point (R_1, B_1) so that $v(G_1) = -u'_1/2$; while G_2 has no saddlepoint, but the standard solution gives $v(G_2) = (-10/3)u'_2$.

The "average" game $G^*(p)$ is obtained if the two sides both receive no special information on the outcome of the chance move:

$G^*(p)$

	B_1	B_2
R_1	$pu'_1(-1/2) + (1-p)u'_2(-7/2)$	$pu'_1 + (1-p)u'_2(-3)$
R_2	$pu'_1(-5/4) + (1-p)u'_2(-13/4)$	$pu'_1(-1/2) + (1-p)u_2(-7/2)$

For convenience let $a = pu'_1$ and $b = (1-p)u'_2$.

The value of $G^*(p)$ is given by the following formula unless there is a saddlepoint:

$$v^*(p) = \frac{[(a + 7b)^2 + (a - 3b)(5a + 13b)]/4}{[-a - 7b + a/4 + 25b/4]}$$

$$= -\frac{2}{3}\left[\frac{3a^2 + 6ab + 5b^2}{a + b}\right]$$

Since the matrix of G^* as a function of a and b is:

$$\begin{Vmatrix} -(a/2 + 7b/2) & a - 3b \\ -(5a/4 + 13b/4 & -(a/2 + 7b/2) \end{Vmatrix},$$

when $b = 0$, there is a saddlepoint (R_1, B_1) corresponding to the solution of G_1. To generalize we must ask: What values of a and b are feasible and satisfy $-(5a/4 + 13b/4) \leq -(a/2 + 7b/2) \leq a - 3b$? The other entries in the matrix clearly can never (i.e., for no $a, b > 0$) be saddlepoints. The necessary and sufficient condition is: $b \leq 3a$. Therefore, the complete solution for $G^*(p)$ is as follows:

$$v^*(p) = \begin{cases} -2(3a^2 + 6ab + 5b^2)/3(a+b) & \text{if } b \geq 3a \\ -(a/2 + 7b/2) & \text{if } b \leq 3a. \end{cases}$$

As shown in the previous paper, the I-game need not be adequately represented by the average game G*(p). Red may take advantage of the fact that he knows his mission type: the so-called chance move at the top of the diagrams is, of course, a fiction which

is designed to take account of Blue's willingness to choose his strategies by Bayesian techniques with the prior probabilities (p, 1-p). Exploration of Red's potential advantages requires that we analyze the four strategy choices R_{11}, R_{12}, R_{21}, R_{22} where : R_{ij} = "choose R_i if G_1, and R_j if G_2 is the 'true' game." The game matrix for this 4 x 2 strategic conflict model together with the six 2 x 2 subgame matrices are given in Table 1 below. All utilities have been multiplied by -1 in this table.

Table 1

$(-1)\ \overline{G}(p)$

	B_1	B_2
R_{11}	$pu_1'(1/2) + (1-p)\,u_2'(7/2)$	$-pu_1' + 3\,(1-p)\,u_2'$
R_{12}	$pu_1'(1/2) + (1-p)\,u_2'(13/4)$	$-pu_1' + (1-p)\,u_2'\,(7/2)$
R_{21}	$pu_1'(5/4) + (1-p)\,u_2'(7/2)$	$pu_1'(7/2) + 3\,(1-p)\,u_2'$
R_{22}	$pu_1'(5/4) + (1-p)\,u_2'(13/4)$	$pu_1'(1/2) + (1-p)\,u_2'(7/2)$

	B_1	B_2		Formula for Value of Subgame When There is No Saddlepoint
R_{11}	$a/2 + 7b/2$	$-a + 3b$	v_{12}^{11}	$= \dfrac{(a+7b)\,(-2a+7b) - (2a+13b)\,(-a+3b)}{(0.a + 3b/4)\,4}$
R_{12}	$a/2 + 13b/4$	$-a + 7b/2$		
R_{11}	$a/2 + 7b/2$	$-a + 3b$	v_{21}^{11}	$= \dfrac{(a+7b)\,(7a+6b) - (5a+14b)\,(-a+3b)}{15a + 0.b}$
R_{21}	$5a/4 + 7b/2$	$7a/2 + 3b$		
R_{11}	$a/2 + 7b/2$	$-a + 3b$	v_{22}^{11}	$= \dfrac{(a+7b)^2 + (a-3b)\,(5a+13b)}{3a + 3b}$
R_{22}	$5a/4 + 13b/4$	$a/2 + 7b/2$		
R_{12}	$a/2 + 13b/4$	$-a + 7b/2$	v_{22}^{12}	$= \dfrac{(2a+13b)\,(a+7b) + (2a-7b)\,(5a+13b)}{(6a + 0.b)}$
R_{22}	$5a/4 + 13b/4$	$a/2 + 7b/2$		
R_{12}	$a/2 + 13b/4$	$-a + 7b/2$	v_{21}^{12}	$= \dfrac{(2a+13b)\,(7a+6b) + (2a-7b)\,(5a+14b)}{30a - 6b}$
R_{21}	$5a/4 + 7b/2$	$7a/2 + 3b$		
R_{21}	$5a/4 + 7b/2$	$7a/2 + 3b$	v_{22}^{21}	$= \dfrac{(5a+14b)\,(a+7b) - (5a+13b)\,(7a+6b)}{-24a + 6b}$
R_{22}	$5a/4 + 13b/4$	$a/2 + 7b/2$		

All six of the possible 2×2 subgames must be solved in order to find the correct solution of the 4×2 game. The minimum of their six separate values is the value of the game $\overline{G}(p)$. Of course, the minimum may shift about as p, u_1' and u_2' are varied, so that we must also take into account the conditions under which the minimum value is attained at a particular subgame. We shall express these conditions in terms of a and b for the present purposes. Table 2 gives the possible saddlepoints. The subgames are listed as G_{ij}^{kl} according to the pair of Red strategies R_{kl}, R_{ij} selected.

Table 2

Subgame	Conditions for Saddlepoint	Saddlepoint
G^{11}_{12}	$-a + 7b/2 \leq a/2 + 13b/4 \leq a/2 + 7b/2$ if $b \leq 6a$	(R_{12}, B_1)
G^{11}_{21}	$-a + 3b \leq a/2 + 7b/2 \leq 5a/4 + 7b/2$ for all $a, b > 0$	(R_{11}, B_1)
G^{11}_{22}	$-a + 3b \leq a/2 + 7b/2 \leq 5a/4 + 13b/4$ if $b \leq 3a$	(R_{11}, B_1)
G^{12}_{22}	$-a + 7b/2 \leq a/2 + 13b/4 \leq 5a/4 + 13b/4$ if $b \leq 6a$	(R_{12}, B_1)
G^{12}_{21}	$-a + 7b/2 \leq a/2 + 13b/4 \leq 5a/4 + 7b/2$ if $b \leq 6a$	(R_{12}, B_1)
G^{21}_{22}	$a/2 + 7b/2 \leq 5a/4 + 13b/4 \leq 5a/4 + 7b/2$ if $b \leq 6a$	(R_{22}, B_1)

Now we can compare the six values of the subgames and seek out the maximum.

Table 3

Subgame	Value	Conditions
G_{12}^{11}	$-a/2 - 13b/4$	$b \leq 6a$
	$-10b/3$	$b > 6a$
G_{21}^{11}	$-a/2 - 7b/2$	all $a, b > 0$
G_{22}^{11}	$-a/2 - 7b/2$	$b \leq 3a$
	$-[2(a+b) + 4b^2/3(a+b)]$	$b > 3a$
G_{22}^{12}	$-a/2 - 13b/4$	all $a, b > 0$
G_{21}^{12}	$-a/2 - 13b/4$	$b \leq 6a$
	$-\frac{2}{3}[(a+5b) + a^2/(5a-b)]$	$b > 6a$
G_{22}^{21}	$-5a/4 - 13b/4$	$b \leq 6a$
	$-\frac{2}{3}[(2a+5b) + a^2/(4a-b)]$	$b > 6a$

The maximum of the six values is easily found by examination of the signs of all possible differences, and gives the following function of a and b for the value of $\overline{G}(p)$:

$$\overline{v}(p) = \begin{cases} -a/2 - 13b/4 & \text{if } 0 < b \leq 6a \quad (G^{12}_{22} \text{ or } G^{12}_{21}) \\ -10b/3 & \text{if } 6a < b \quad (G^{11}_{12}) \end{cases}$$

It will be seen now that $\overline{v}(p) > v^*(p)$ for all a and b, and therefore this holds for all p regardless of the values of the utilities: u_1' and u_2'. The decision as to which game, $\overline{G}(p)$ or $G^*(p)$ to play belongs to Red who possesses the advantage in knowing which is the true game (G_1 or G_2). As the maximizer, he naturally prefers $\overline{G}(p)$. Blue can make the same calculations and so he can and should hold Red to the value of the game $\overline{G}(p)$; any other Blue strategy results in a worse outcome for Blue. Optimal strategies in terms of parameters a and b are:

$$\text{(i) if } 0 < b \leq 6a : \quad R^* = \begin{cases} R_1 \text{ if true game is } G_1 . \\ R_2 \text{ if true game is } G_2 \end{cases}$$

$$B^* = B_1$$

$$\text{(ii) if } b > 6a : \quad R^* = \left(\frac{b-6a}{3b}, \frac{6a+2b}{3b}\right)$$

$$B^* = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Note that Red's mixed strategy in case (ii) refers to the pure strategies R_{11} and R_{12}, not to R_1 and R_2. The mixed strategy can be interpreted: if G_1 is the true game, choose R_1 (with probability one), but if G_2 is the true game, choose $\begin{Bmatrix} R_1 \text{ with probability } \frac{b-6a}{3b} \\ R_2 \text{ with probability } \frac{6a+2b}{3b} \end{Bmatrix}$.

CHAPTER VIII

FORMULATION OF A GAME OF SEARCH AND PURSUIT

John P. Mayberry
Francis M. Sand
Alan J. Truelove

FORMULATION OF A GAME OF SEARCH AND PURSUIT

I. INTRODUCTION

In this paper we formulate a two-stage game, whose first stage is a search game (in which very little information is available to the antagonists), and whose second stage is a pursuit game (which is a differential game of perfect information). The combined game is intended to represent aspects of realistic ASW situations, where a search may be followed (in case the searcher is successful in locating the evader) by a pursuit; or where a submarine, with limited range and endurance while submerged, must try to progress towards his objective -- without making his surfacing point too predictable.

Of course, the behavior of both the pursuer P and the evader E, during the first stage, will be influenced by their knowledge that the second stage will be played at some future time. The nature and extent of that influence may be little or great, depending on the parameters of the game (principally the ratio w of E's speed to P's speed, and the radius r of P's capture-circle), on the initial conditions (principally the initial positions of E and P, relative to E's destination), or the nature of E's destination (which we assume, throughout this paper, to be a straight line ℓ, called the "life-line"), and on the "re-detection time," when stage 1 ends and stage 2 begins (which may be random, or may occur when one or the other of P or E crosses some given curve). Several interesting variants will be suggested below.

In section 2 below, we formulate the two-stage game more precisely; in section 3 we present in some detail the solution to the second stage (given the initial conditions to that stage), for the special case where the capture-zone is negligibly small, so that P must contact E to capture; in Section 4, we describe how we intend to proceed in order to solve the first stage (and thus to solve the whole game); and in section 5, we describe how the results of these two-stage games could be of practical value in certain ASW situations.

2. FORMULATION

We may guide our intuition by thinking of a submarine as the evader E, a destroyer as the pursuer P, and a sub-pen area as the life-line ℓ. At time $t = 0$, when P and E each discover the other's location, the sub dives and both proceed quietly (neither detectable by the other) until redetection occurs. At that time, stage 2 begins; both proceed at top speed (in full awareness of each othe[illegible]ition) to see whether or not P can capture E before E reaches ℓ.

This explanation implies that the absolute speeds of both P and E are greater in stage 2, but it is only the ratio of the speeds which enters into the mathematics of the problem; we simplify matters by assuming that the ratio w of E's speed to P's speed does not change when we go from stage 1 to stage 2.

We quote from the description of the lifeline game in [DG], changing to our own notation:

"The two points P and E have each simple motion* in a half-plane bounded by a line ℓ. Their speeds are parameters and as only the ratio matters, we shall let P have speed unity and E, speed $w (w \gtreqless 1)$.

---

* i.e., each moves at a fixed, preassigned speed, and each has complete control over his own direction of motion at any time.

Capture...is to occur when the distance $PE < r$ ... E's objective is to reach ℓ prior to capture and, naturally, P's objective is to prevent him ."

To complete the description of our two-stage game, we need only add these conditions:

At $t = 0$, they know each other's locations; for $0 < t < t_1$, neither gains more information; at $t = t_1$, stage 1 is over; for $t > t_1$, stage 2 is played, starting from the locations of the participants at $t = t_1$, with perfect information. Of course, the final outcome of stage 2 (viz. whether E escapes or is captured) will be completely determined by the players' locations at $t = t_1$, in a way which will be made explicit in section 3 below.

Now let us present some preliminary remarks:

(i) If $w > 1$, the evader can always escape in stage 2 -- unless capture occurs at $t = t_1$. Because of this exception, we did not wish to ignore the case $w > 1$.

(ii) If the initial locations at $t = 0$ would have permitted E to escape when P could see and pursue him, then the same action by E will result in escape when P cannot see him.

(iii) If $w > 1$, then E could escape without stage 1; then, from (ii) above, we see that E can escape anyway.

(iv) If t_1 is small, then the initial interval represented by stage 1 of the game will have little influence on the outcome.

(v) If t_1 is sufficiently large, E can reach ℓ before t_1 and will escape.

(vi) When $t_1 = 0$, this 2-stage game becomes the lifeline game; ref [DG] and section 3 describe how to determine the capture zone CZ and escape zone EZ.

(vii) In general, optimal play in stage 1 will involve mixed strategies, so we will be forced to compute probability of escape rather than EZ and CZ. Probability of escape P_e will depend on initial locations P and E, on the speed-ratio w, on the capture-radius r, and on the first-stage duration t_1.

(viii) For $t_1 > \frac{\overline{E\ell}}{w}$, $P_e = 1$.

(ix) If P and E are such that E is in EZ (P, w, r), then $P_e(P,E,w,r,t_1) = 1$.

(x) $P_e(P,E,w,r,t_1) \leq P_e(P,E,w,r,t_1')$ if $t_1 < t_1'$; in other words, the longer the "blind-search" period, the better E's chances of escaping.

(xi) If w = 1, a special problem arises; the second stage is then a simpler, but different, problem.

3. ANALYSIS OF THE SECOND STAGE AS A LIFE-LINE GAME

E is trying to reach the destination line ℓ; P is trying to intercept E at or before the time E reaches the barrier.

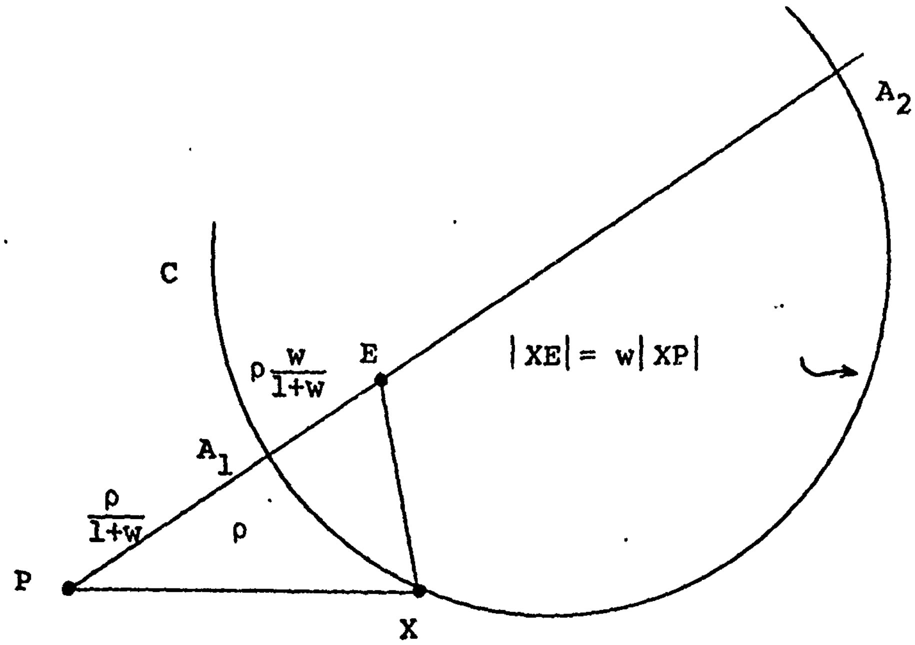
At time $t = t_1$, let the location of the two players with respect to ℓ be shown in Figure 1. E will be taken as the origin of coordinates (0,0); E and the line P (y = -d) will, for the moment, be regarded as fixed, and the position of P will be denoted by $(\rho \cos \vartheta, \rho \sin \vartheta)$, assumed variable.

At all later times $t > t_1$, both P and E know each other's position. Recall that

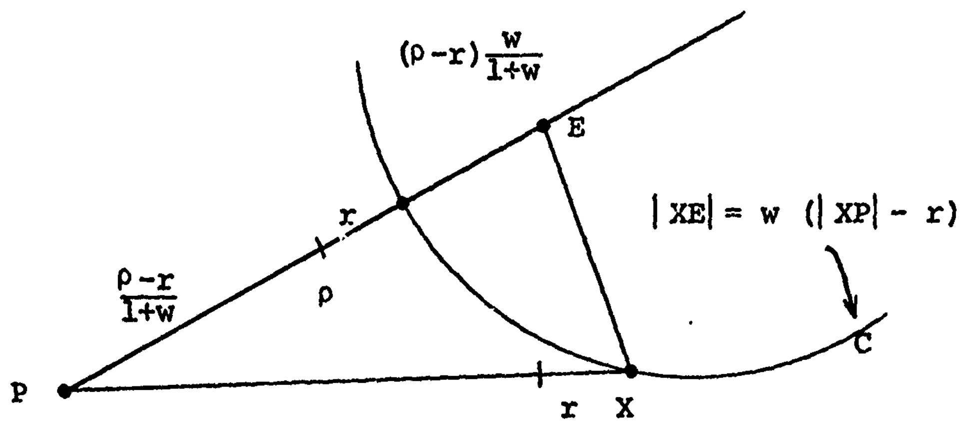
$$\left(\frac{\text{E's speed}}{\text{P's speed}}\right) = w,$$

(which we now will assume < 1 because of remarks (iii), (xi), above).

We present explicit solutions in the case r=0, when P must actually meet E in order to effect capture. As reference [DG] shows, the case r > 0 is qualtitatively similar in many respects, but the explicit solutions would certainly be much more complicated.



(a) Case r=0: The circle of Appolonius



(b) Case r > 0: A curve of the Fourth Degree (Biquadratic)

Figure 1

Boundary curves between capture and escape (Stage 2)

The locus C of points X, such that E and P could reach X in the same length of time, is a circle, the circle of Appolonius; see Figure 1(a). In the case $r > 0$, the circle is replaced by a biquadratic curve C of similar appearance; see Figure 1(b). The points inside C (including E itself) are points which E could reach sooner than P could; the points outside C (including P itself) are points which P could reach sooner than E could. A diameter of C is identified by the two points A_1 and A_2 on the line PE which satisfy

$$\frac{|A_1E|}{|A_1P|} = \frac{|A_2E|}{|A_2P|} = w .$$

If C does not meet ℓ, then P can block E by heading for the point of ℓ closest to C; P would reach that point before E could. Of course, as E and P move, the circle C may also move; but C cannot move so as to meet ℓ unless P makes a mistake; and as long as C does not meet ℓ, P can remain "below" E, moving upward to capture at some later time.

If C meets ℓ at two points, then E can escape from P by heading for a point of ℓ between them; E will arrive there first, and escape. (See remark (ii) above.)

Considering E fixed and P variable, the locus of points P, such that P would just barely capture E, is the locus of points P, such that the circle of Appolonius C is tangent to ℓ, as shown in Figure 2. That locus consists of the boundary of a half-ellipse, as shown in Figure 3. We call the interior of that half-ellipse the capture-area CA, since it consists of possible locations for P where capture would certainly result if both antagonists behaved optimally; its exterior, consisting of possible locations for P where escape would certainly result if both antagonists behaved optimally, is called the escape area EA. Note that the CA and the EA are in the space of locations for P.

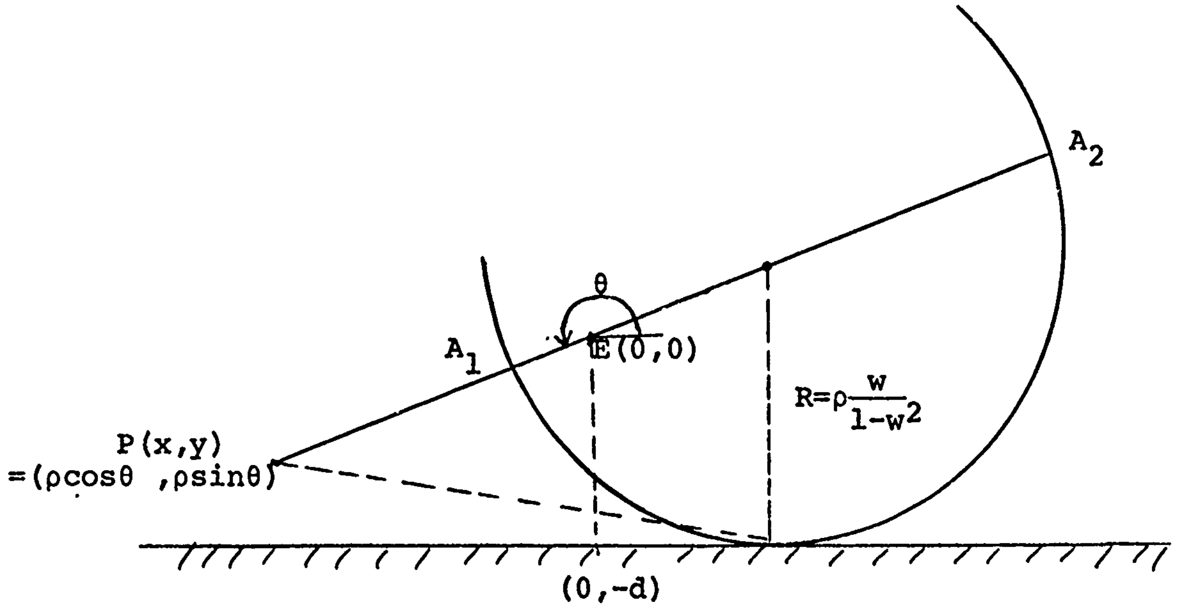


Figure 2

Apollonius' Circle, Tangential to Destination Line ℓ

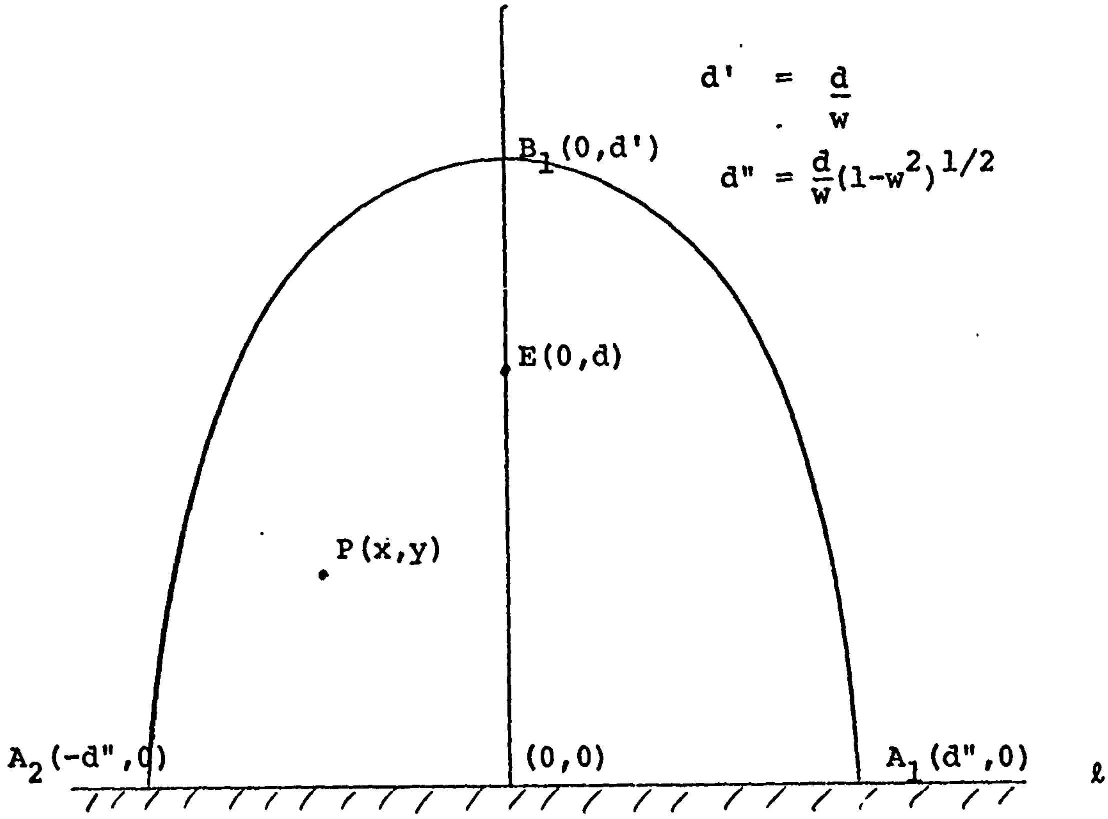


Figure 3

The capture-area CA and the Escape-area EA in the space of locations for P.

$$\left[\frac{y}{d'}\right]^2 + \left[\frac{x}{d''}\right]^2 \leq 1 \quad \text{implies capture ;}$$

$$(1-w^2)\left[\frac{y}{d}\right]^2 + \left[\frac{x}{d}\right]^2 \leq \frac{1-w^2}{w^2} .$$

Now consider P fixed, and E variable. Now the set of points where E could be located, and still escape, is the outside of one limb of a hyperbola, as shown in Figure 4 . That region is the escape zone, EZ. The "inside" of that hyperbola, including, of course, all points directly above P, is the set of points where E could be captured; it is the capture-zone CZ. Note that the EZ and the CZ are in the space of locations for E.

The condition for eventual capture, assuming optimal play in stage 2, is that

$$\left(w \cdot \frac{y+d}{d}\right)^2 + \left(\frac{x}{d}\right)^2 \left(\frac{w^2}{1-w^2}\right) \leq 1,$$

at the beginning of stage 2, where d is the distance of E from ℓ , y is the distance of P from ℓ , and x is the "horizontal" component* of the separation between E and P.

We define the function capt (x,y,d) by the relation:

$$\text{capt}\ (x,y,d) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{if } (1-w^2)\left(\frac{y}{d}\right)^2 + \left(\frac{x}{d}\right)^2 - \frac{1-w^2}{w^2} \begin{Bmatrix} >0\ , \\ \leq 0\ . \end{Bmatrix}$$

Thus capt (x,y,d) = 1 if,and only if, capture can be forced by P in stage 2.

*(i.e., parallel to ℓ)

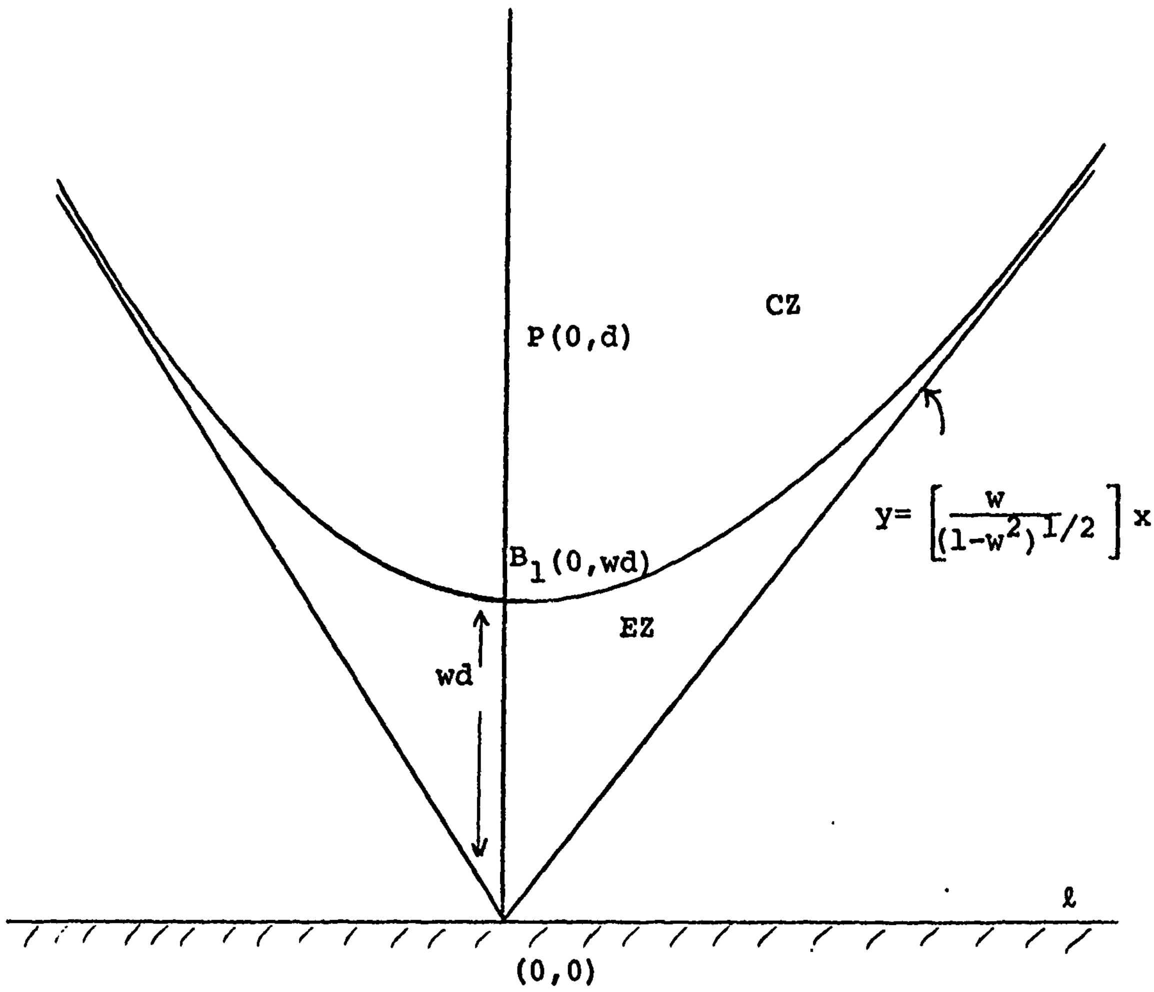


Figure 4

The Capture-Zone CZ and the Escape-Zone EZ in the space of locations for E.

4. APPROACH TO SOLUTION OF STAGE 1

Using the results of section 3 above, we can predict with certainty whether or not a given system configuration (i. e., the locations of E and P relative to ℓ), at the beginning of stage 2, would result in capture of E by P during stage 2, assuming optimal action by both in sate 2. The formulae for making that prediction were supplied explicitly in section 3 for the case $r=0$; the case $r > 0$ is qualitatively similar (although it favors P , and is therefore quantitatively not correct).

Let us now consider stage 1 as a game, whose payoffs are determined by the locations of P and E at the end of state 1. We shall first consider the game of kind, where the payoffs are the logical alternatives: "E wins", or "P wins". We will identify P as the maximizing player, and say that the "payoff" (meaning, the payoff from E to P) is $= 1$ if capture can be guaranteed from the system configuration at the end of state 1, and is $= 0$ if escape can be guaranteed from the system configuration at the end of stage 1. (The analysis of stage 2 as a pursuit game shows that one or other of these conditions must hold.)

Then P's goal is to maximize the expected value of this numerical payoff —or, what is the same thing, the probability of capture— while E's goal is exactly opposite.

The strategies which are open to each of P and E comprise a whole function space of paths which each might follow, during the time $0 \leq t \leq t_1$, of stage 1. However, it is obvious that the path followed

by either is totally irrelevant; the only things that matter are the end-points--their locations at time $t = t_1$ when re-detection occurs. Thus we can identify the pure strategies for each player with the set of points at which they could be located at time t_1; in each case, this will be a circular disk, centered at his original ($t = 0$) position (E_o or P_o, as the case may be), omitting any part of the disk which might be on the wrong (inaccessible) side of $\mathcal{l}$. Recall that P's velocity was taken as 1, and E's as w; thus, the pure strategies of P correspond to a circular disk $\mathcal{D}_P$ centered at P_o and with radius t_1, while the pure strategies of E correspond to a circular disk $\mathcal{D}_E$ centered at E_o and with radius wt_1, each restricted to the accessible side of $\mathcal{l}$ (see Figure 5).

Now, although all those strategies are legal, not all are plausible; simple arguments will further restrict the set of rational strategies for each player. In particular, note, from Figures 3 and 4, that it will never profit either player, no matter where the other is, to be farther from $\mathcal{l}$ and opposite the same point of $\mathcal{l}$. It follows that, in determining what constitutes rational play in this game of kind, we need only consider the points of those two cir[illegible] disks which are not separated from $\mathcal{l}$ by other points of the sa[illegible]sks--i,e., the semi-circles bounding the disks on the side towards $\mathcal{l}$. (If a disk intersected $\mathcal{l}$, then the set of "nearest points" would possess a linear portion between two circular arcs.) Those semi-circles, whose points now represent plausible strategies, are labeled $\mathcal{S}_P$ and $\mathcal{S}_E$ on Figure 6. The points X of those semi-circles, and thus the strategies, can be identified by the angle between XP_o (or XE_o) and $\mathcal{l}$; a pure strategy for P is denoted [illegible]y α, and a pure strategy for E by β; the resulting positions of P and E [illegible]ectively, at time $t = t_1$, will be denoted by $P_1(\alpha)$ and $E_1(\beta)$.

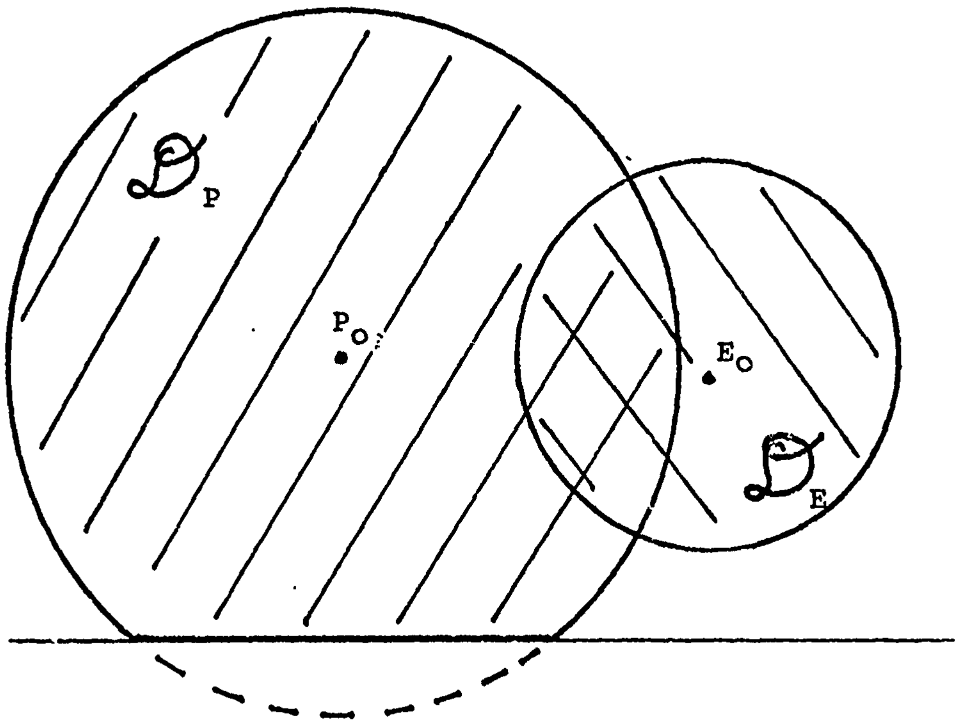


Figure 5

Feasible strategies in Stage 1

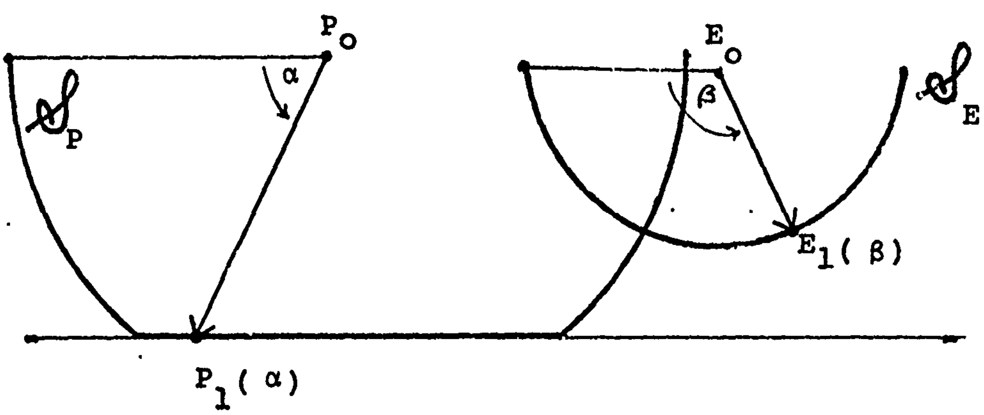


Figure 6

Plausible Strategies in Stage 1.

To recapitulate: this stage 1 game, denoted by Γ, has players P and E; P has pure strategies α, $0 \leq \alpha \leq \pi$; E has pure strategies β, $0 \leq \beta \leq \pi$; and the payoff p (α, β) is defined by

$$p(\alpha, \beta) = \text{capt}(x, y, d),$$

where

x = x-coord. of P_1 (α) - x-coord. of E_1

y = y-coord. of P_1 (α) ,

d = y-coord. of E_1 (β) .

The we must ask, "Is this game Γ strictly-determined?" -- i.e., "Is there a way P can be sure of capturing, or a way E can be sure of escaping?" First of all, if $\mathscr{A}_E$ meets ℓ then E can always escape before stage 2 starts. Secondly, if E_o, P_o, w and r are such that E could escape in a (stage-2) pursuit game whose starting positions were E_o and P_o, then E could in Γ follow the same path, P could accomplish no more in Γ (where he has less information), and E will surely escape in Γ. So much is obvious; but the converse is also true: if E has a course of action in Γ which will surely result in escape, but which does not allow E to reach ℓ before t_1, then that course of action would result in escape even if P knew of it, and E would, therefore, escape in the pursuit game (i.e., even if t_1 had been = 0).

At the opposite extreme, P might be able to force capture. That will be possible if there is some point $P^o \in \mathscr{A}_P$ whose capture-zone CZ includes the whole of $\mathscr{A}_E$. The existence of such a point P^o -- or its location, if one does exist -- cannot always be guessed. See Figure 7 for an example.

In each of these cases -- where the probability of escape is either 0 or 1 -- it is reasonable to consider the associated <u>game of degree</u>, where a new payoff is introduced, depending on the length of time before

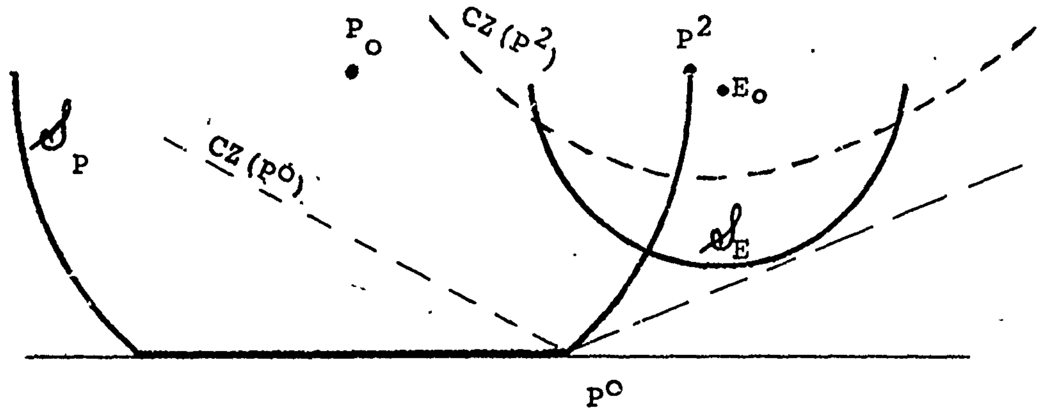


Figure 7

Example of an Optimal Strategy P^O for P

$(w = 1/2)$

capture (or before E reaches $\mathscr{L}$, as the case may be). In this way we can remove the ambiguity in the <u>action</u> recommendations which often characterizes a differential game of kind. We must be careful, in considering the game of degree Γ' associated with a strictly-determined game of kind Γ, not to allow the players in Γ' <u>all</u> the strategies of Γ; the player who can win Γ will, in general, be forced to avoid certain of his strategies in Γ, and he must avoid those strategies in Γ' also. In the example of Figure 7, P is restricted (if he is to win the game of kind) to strategies very near to P^o; if he wished to minimize time to capture, he might consider a direct chase (from P_o towards E_o), but would then find himself near P^2 at time t_1, and E would escape.

Solution of the game of kind, in cases where neither E nor P can force a win, requires consideration of mixed strategies on each side. Simple geometric arguments will probably not suffice to determine the nature of these mixed-strategy solutions.

5. APPLICATION OF THIS TWO-STAGE GAME TO REAL PROBLEMS OF ASW

This game, as described in section 2, is not very close to any realistic ASW situation. Nevertheless, there are a number of directions in which modifications could be made which would result in valuable insights towards actual ASW problems.

(a) We might consider a situation where t_1 was not known with certainty, but where a probability-distribution of redetection times was known to both P and E.

(b) We might consider that, at time t_1, one or both of P and E might be <u>uncertain</u> of redetecting the other -- i.e., each would have only a probability of detecting the other.

(c) Those redetection probabilities might depend on the range (distance between P_1 and E_1).

(d) We might have repetitive intervals of undetectability, with mutual detection between.

(e) Using the general notion of a game of incomplete information (I-game), as in Chapters VI and VII of this report, we may include mutual uncertainty about speeds, objectives (shape and location of Destination - set $\mathcal{C}$), and detection-capabilities.

Reference:

[DG]: "Differential Games", by Rufus Isaacs; J. Wiley & Sons (1965) pp. 257-260.